SMILE WORKCARDS

Properties of Number Pack Three

Contents

Turn the tables

Smile **1394**

You will need some copies of the multiplication square

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

This card is about the numbers which appear in an ordinary multiplication square.

1. 48 appears 4 times. Colour each 48 in red on a multiplication square. *Write* 4 x 12 = 48
6 x 8 = 48
8 x 6 = 48
12 x 4 = 48

Find some other numbers which appear several times.
Use different colours to shade them. Write the multiplication facts.

2. 36 appears 3 times.
Which numbers appear an odd number of times?
Shade each one a different colour on another multiplication square.
What is special about these numbers?

3. The patterns from question (1) and (2) should be symmetrical.
Draw the line of symmetry.
Explain why the multiplication table is symmetrical.

4. Write a list of all the numbers below 50 which do not appear in the table.
Write any comments about these numbers.

Multiplication table patterns

You will need a multiplication square and a calculator

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

Get a multiplication square or draw one yourself.
If you draw it yourself make sure it's correct.

1. Can you see this rectangle of numbers in the multiplication square?

18	24	30
21	28	35

The rectangles below have all been taken from the multiplication table. Copy them and fill in the missing numbers.

6
8

16	
	36

				36
18				

		6
8		12

	36
40	

Turn over

2. (a) Find this 3 x 3 square.

8	9	10
16	18	20
24	27	30

Add the 4 shaded numbers

9+16+20+27 = ■

Add the 4 corner numbers

8+10+24+30 = ■

Multiply the centre number by 4

18 x 4 = ■

What happens?

Choose 2 more 3 x 3 squares and see if the same pattern works.

(b) Find this 4 x 4 square.
Add the 4 shaded numbers.

7+18+18+32 = ■

6	7	8	9
12	14	16	18
18	21	24	27
24	28	32	36

There are other sets of 4 numbers with the same sum in this square.
How many can you find?

Choose another 4 x 4 square and see if the same pattern works.

(c) Choose some larger squares and some rectangles.
Investigate sets of 4 numbers with the same sum.

3. Choose a rectangle.

15	20	25	30
18	24	30	36
21	28	35	42

Multiply the numbers at opposite corners:

15 x 42 = ■
30 x 21 = ■

Investigate this pattern with other rectangles.
Does the pattern **always** work?
Can you explain?

Prime Factors

2 x 2 x 3 x 5 = 60

2 x 2 x 3 = 12

3 x 3 x 3 = 27

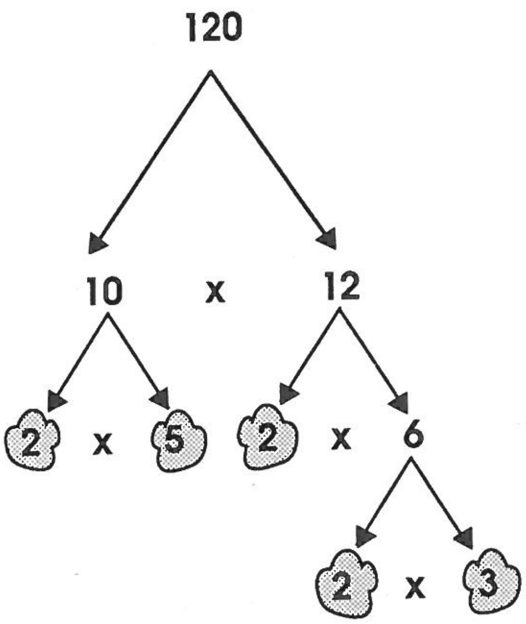
2 x 3 x 3 x 5 = 90

3 x 5 = 15

60, 12, 27, 90 and **15** have each been made by multiplying prime numbers.

Make the following by multiplying prime numbers
54, 63, 72 and **100**

Turn over if you need some ideas about how to approach these problems.



120 ÷ (2) = 60

60 ÷ (2) = 30

30 ÷ (2) = 15

15 ÷ (3) = (5)

120 = 30 x 4

= (3) x 10 x (2) x (2)

= (3) x (2) x (5) x (2) x (2)

2 x 2 x 2 x 3 x 5 = 120

2, 3 and **5** are the **prime factors** of **120.**

Number Names

Each of these describes only **one** number less than 100. *What are they?*

A is an even number
a triangle number
a multiple of 9

B is an odd number
a square number
a factor of 100

D is a prime number
an odd number
a factor of 52

C is a triangle number
an odd number
a multiple of 9

G

F is less than 12
not a triangle number
not a square number
not prime

E is a triangle number
a square number
an even number

Write a description for **G** (you choose the number). *Give it to a friend to solve.*

You will need the micro program **Numbers**

Diagonal Multiples

7 columns boxed in multiples of 3 give this diagonal pattern.

What other multiples will give diagonal patterns ?

1	2	**3**	4	5	**6**	7
8	**9**	10	11	**12**	13	14
15	16	17	**18**	19	20	**21**
22	23	**24**	25	26	**27**	28
29	**30**	31	32	**33**	34	35
36	37	38	**39**	40	41	**42**
43	44	**45**	46	47	**48**	49
50	**51**	52	53	**54**	55	56
57	58	59	**60**	61	62	**63**
64	65	**66**	67	68	**69**	70

Investigate which multiples will give diagonal patterns for other numbers of columns.

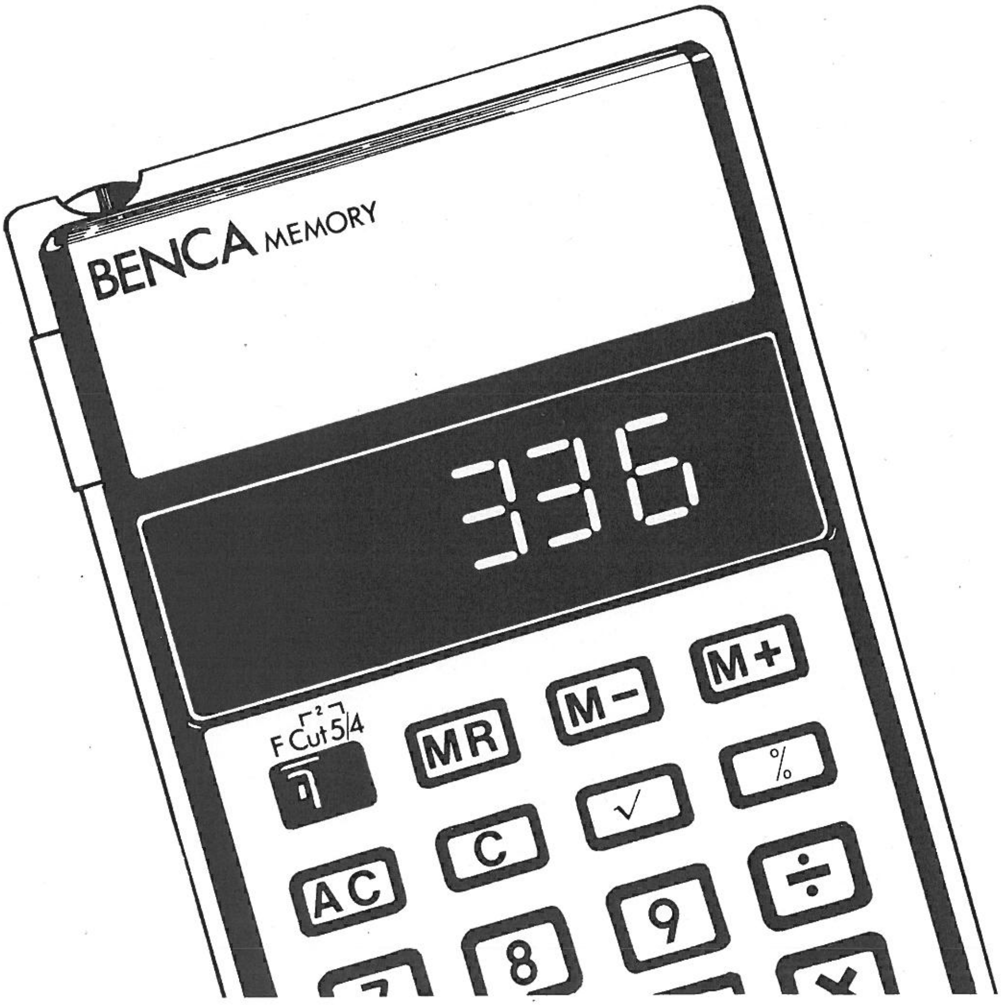
Consecutives

You will need a calculator.

Multiply any 3 consecutive numbers together e.g. $6 \times 7 \times 8$
Is the answer divisible by 24?

Repeat with other sets of 3 consecutive numbers until you can explain why some answers are divisible by 24 and some are not.

(There are hints on the back if you need them.)



SMILE PRESENTS

THE SMITH FAMILY CIRCUS

This pack presents a set of five problems:
Albert Smith, Betty and Colin Smith, Denise Smith, Edward and Freda Smith, and The Last Problem.

It would be sensible to work with a small group. You can share out the cards, but you won't be able to crack the code in the Last Problem until you have solved all the others.

Denise Smith

Denise started with her age and subtracted the sum of the digits. The answer was 36.

She also noticed that when she multiplied the digits and then added on the sum of the digits and then added 8. The answer was her age.

How old is Denise?

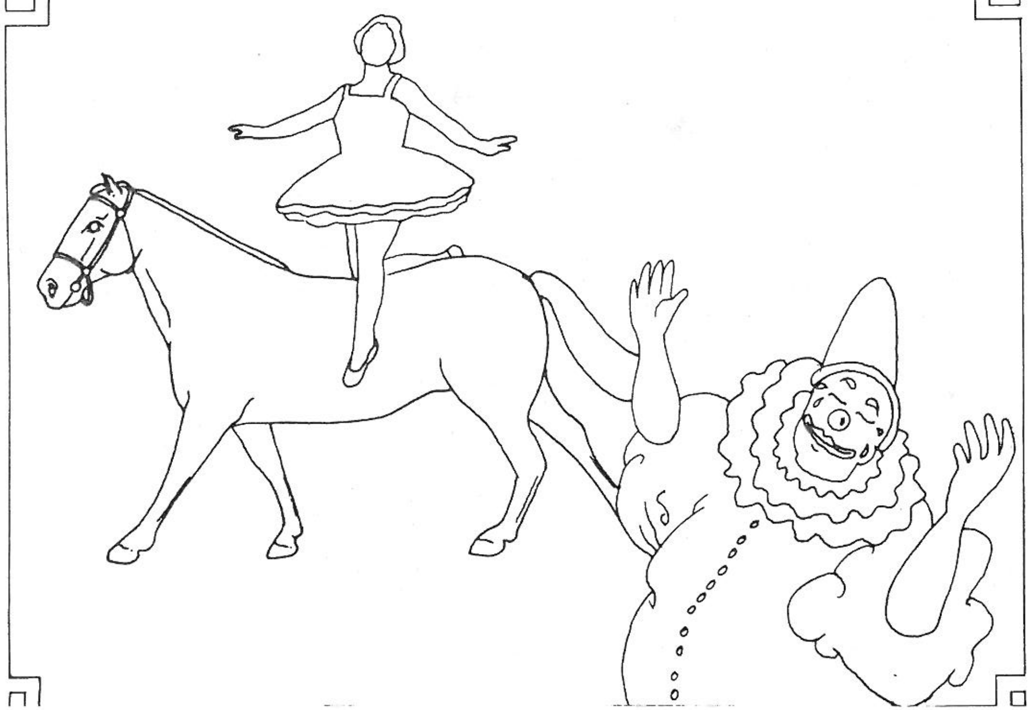
Betty and Colin Smith

Last year Betty's age was a square number. Next year it will be a cube number.

How old is Betty?

Her father,Colin,said that his age was both a square number and a cube number.

How old is Colin?



Edward and Freda Smith

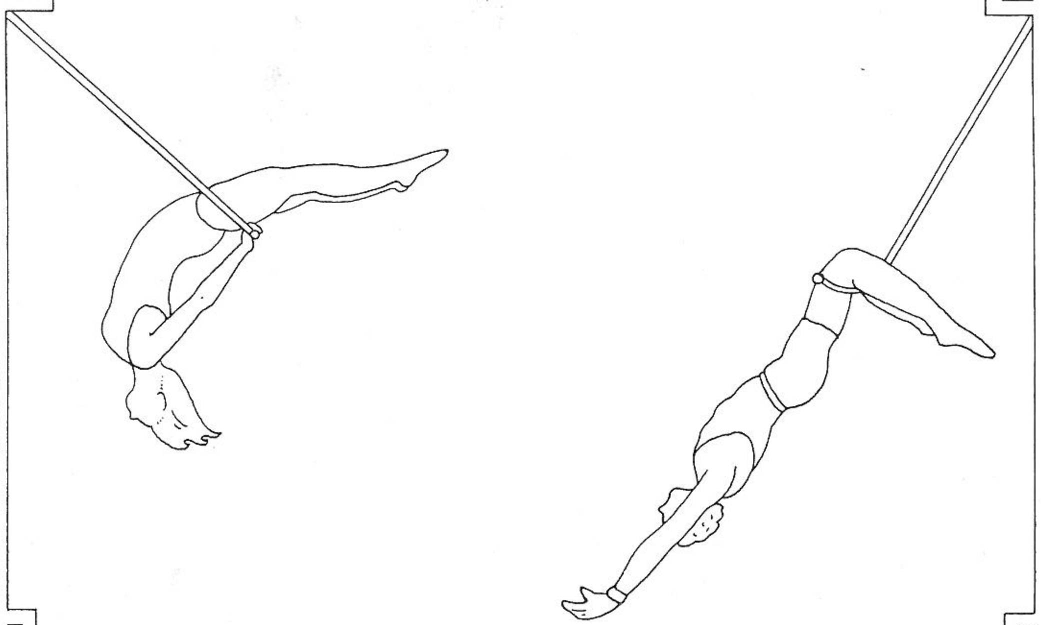
Edward drew a rectangle. The sides all measured a whole number of centimetres.

Edward said that you could work out his age by two different methods: you could calculate the area of the rectangle or the perimeter.

How old is Edward?

Freda is Edward's older sister. She drew a different rectangle. Again, the sides all measured a whole number of centimetres. She also said that the area of her rectangle was equal to the perimeter and that they were both equal to her age.

How old is Freda?

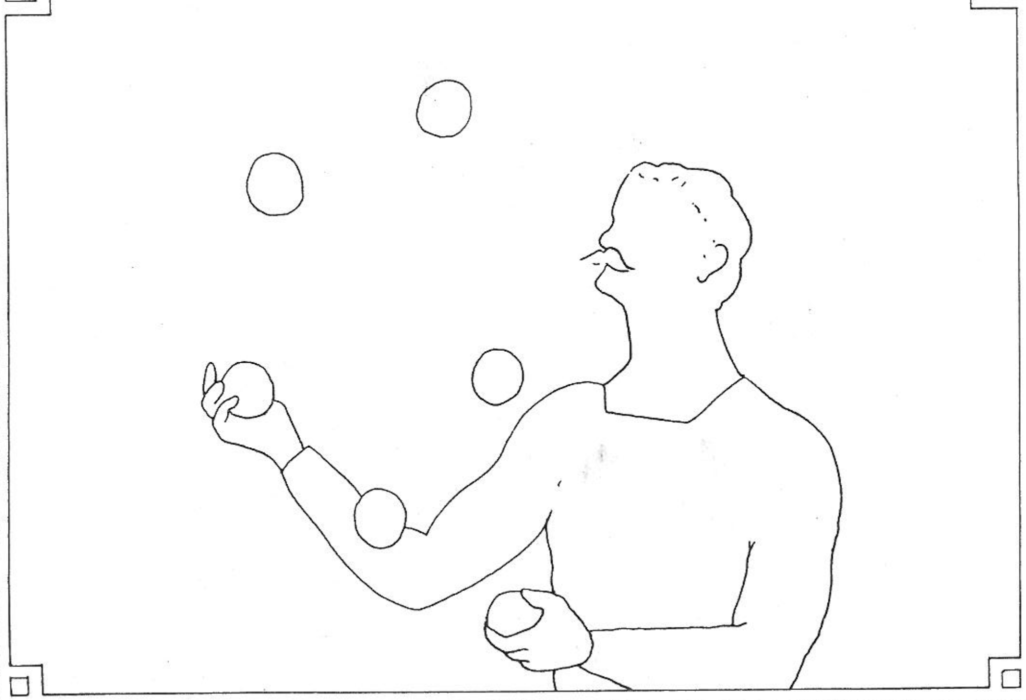


Albert Smith

Albert divided his age by some different numbers. The remainder was always 1.

He divided his age by 2 ⟶ The remainder was 1
He divided his age by 3 ⟶ The remainder was 1
He divided his age by 4 ⟶ The remainder was 1
He divided his age by 5 ⟶ The remainder was 1
He divided his age by 6 ⟶ The remainder was 1

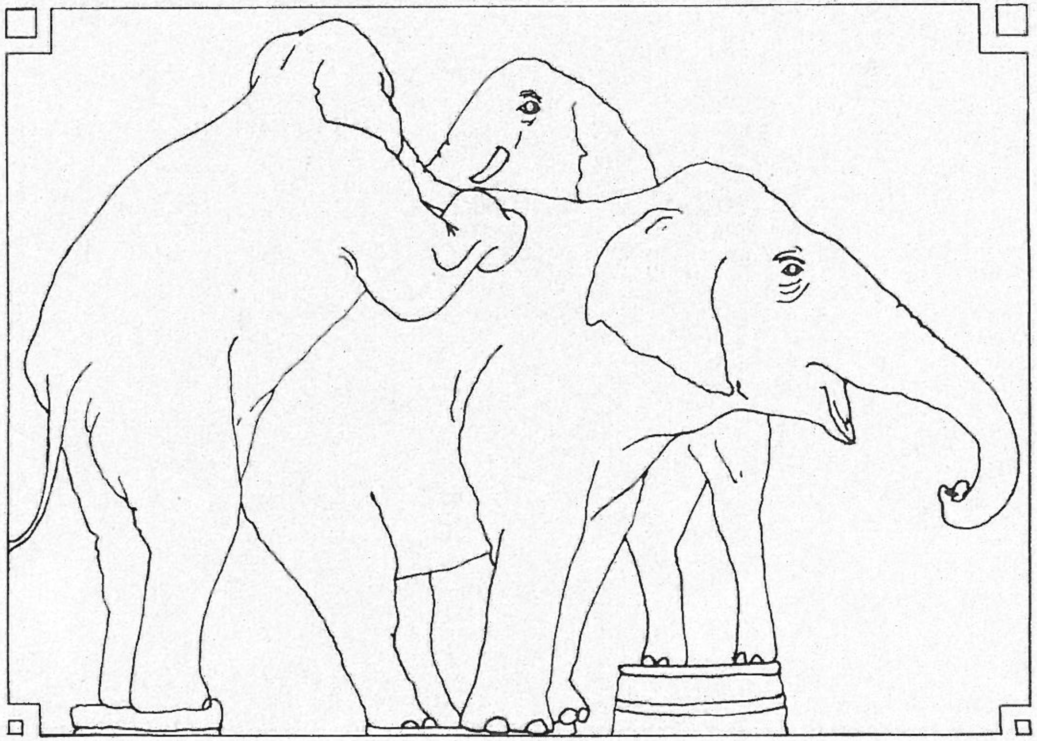
How old is Albert?



The Last Problem

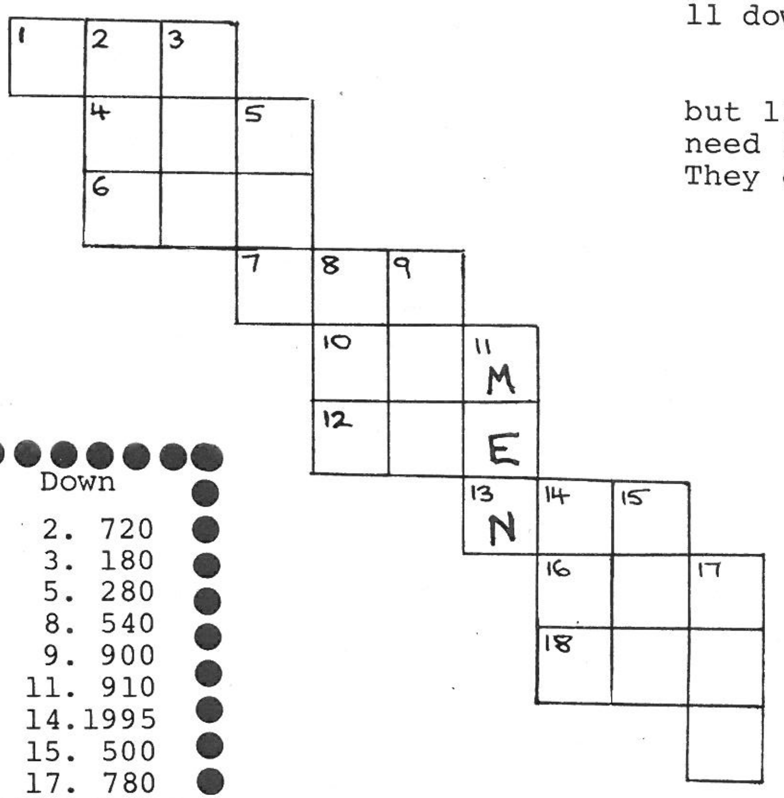
When you have worked out the ages of all the Smith family, you should be able to crack the code:

	A	B	C	D	E	F
G	24	59	8	46	19	28
L	49	69	64	56	89	35
A	53	17	11	94	17	41
S	61	8	12	1	5	3
V	18	41	99	47	75	56
O	81	26	90	25	76	80
T	91	25	50	87	52	97
D	46	88	37	68	84	18
E	23	2	6	70	16	15
W	22	68	20	13	83	38



THE 'TIMES' CROSSWORD

A	1	N	14
B	2	O	15
C	3	P	16
D	4	Q	17
E	5	R	18
F	6	S	19
G	7	T	20
H	8	U	21
I	9	V	22
J	10	W	23
K	11	X	24
L	12	Y	25
M	13	Z	26



Across	Down
1. 180	2. 720
4. 540	3. 180
6. 280	5. 280
7.1200	8. 540
10.2106	9. 900
12. 50	11. 910
13.5880	14.1995
16. 475	15. 500
18. 300	17. 780

The answers in the cross-word below are all proper words. The answer to 11 down is MEN because:-

910 = 2x5x7x13 (prime factors)

but 11 down only has 3 letters, so we need 3 factors.
They could be 2x5x91 or 2x7x65 or 2x13x35 or 5x7x26 or 5x13x14 or 7x10x13

The highest number we can use is 26 (Why?)

So:-
910 = 5x7x26 or 5x13x14 or 7x10x13

E G Z or E M N or G J M

and the only possible word is MEN

Number Challenge

There are **three** numbers . . .

two of the numbers are less than forty . . .

one of the numbers is the square root of one of the other numbers . . .

the three numbers added together are less than a hundred . . .

one of them is a prime number . . .

two of them are even . . .

the three numbers add to make a prime number . . .

the digits of this total add together to make one of the numbers . . .

one of the numbers is one more than a multiple of ten . . .

two of the numbers have more than one digit . . .

none of them are triangle numbers . . .

What are the **three** numbers?

HCF & LCM

HCF

HCF stands for highest common factor . . .
. . . The HCF of 6 and 9 is 3 because 3 is the largest number which divides exactly into both 6 and 9.

LCM

LCM stands for lowest common multiple . . .
. . . The LCM of 6 and 9 is 18 because 18 is the smallest number which 6 and 9 both divide into.

(1) Work out (a) the HCF of 6 and 8.
(b) the LCM of 6 and 8.

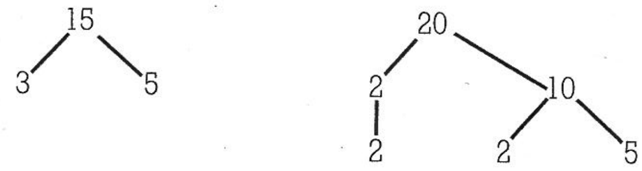
The answers are at the foot of the back page.
If you have got them wrong, make sure you understand HCF and LCM before you continue.

Draw a table to show the HCF of some pairs of numbers.

HFC	1	2	3	4	5	6	7	8	9	12	13	14	18	19	20	21
1																
2																
3																
4																
5																
6								2	3							
7																
8																
9																
12																
13																
14																
18																
19																
20																
21																
22																
23																

Describe any patterns in your table. Explain them if you can.

All numbers can be expressed as the product of prime numbers:

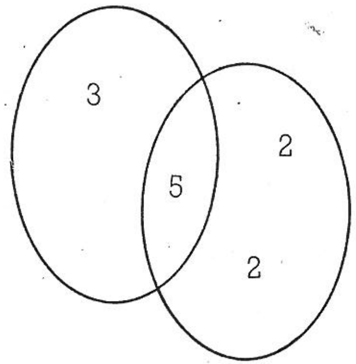


$15=3\times5$

$20=2\times2\times5$

These prime factors can be put into a diagram:

Prime factors of 15



Prime factors of 20

Draw diagrams like this for some other pairs of numbers.

How do the diagrams help to find the HCF and LCM easily?

turn over

Tabulate your results from the previous page:

FIRST NUMBER	SECOND NUMBER	HCF	LCM
15	20	5	60
6	9	3	18
6	8	2	24

What is the relationship between any two numbers and their HCF and LCM?
(You may need to add some more results to your table.)

Explain the rule, if you can – the work on the previous page might help.

ANSWERS FOR Q1: (a) 2 (b) 24

Quadratics and Primes

You may like to use a graphic calculator or spreadsheet.

Prime numbers are of great fascination to mathematicians. Many methods have been developed to generate them.

Quadratics can generate primes, e.g. $x \longrightarrow x^2 - x + 17$

x	$x^2 - x + 17$
0	$0^2 - 0 + 17 = 17$
1	$1^2 - 1 + 17 = 17$
2	$2^2 - 2 + 17 = 19$
30	$30^2 - 30 + 17 = 887$

The quadratic $x \longrightarrow x^2 - x + 17$ generates many prime numbers. However it does not always produce a prime.

- Find some values of x for which this quadratic does not produce a prime.

There is a list of primes on the back of this card to help you check.

$x^2 - x + 41$ also generates primes.

- Does this quadratic *always* produce a prime number?

- Find other quadratics which are good generators of primes.

List of primes

Prime numbers up to 2700

2	229	523	857	1201	1559	1933	2311
3	233	541	859	1213	1567	1949	2333
5	239	547	863	1217	1571	1951	2339
7	241	557	877	1223	1579	1973	2341
11	251	563	881	1229	1583	1979	2347
13	257	569	883	1231	1597	1987	2351
17	263	571	887	1237	1601	1993	2357
19	269	577	907	1249	1607	1997	2371
23	271	587	911	1259	1609	1999	2377
29	277	593	919	1277	1613	2003	2381
31	281	599	929	1279	1619	2011	2383
37	283	601	937	1283	1621	2017	2389
41	293	607	941	1289	1627	2027	2393
43	307	613	947	1291	1637	2029	2399
47	311	617	953	1297	1657	2039	2411
53	313	619	967	1301	1663	2053	2417
59	317	631	971	1303	1667	2063	2423
61	331	641	977	1307	1669	2069	2437
67	337	643	983	1319	1693	2081	2441
71	347	647	991	1321	1697	2083	2447
73	349	653	997	1327	1699	2087	2459
79	353	659	1009	1361	1709	2089	2467
83	359	661	1013	1367	1721	2099	2473
89	367	673	1019	1373	1723	2111	2477
97	373	677	1021	1381	1733	2113	2503
101	379	683	1031	1399	1741	2129	2521
103	383	691	1033	1409	1747	2131	2531
107	389	701	1039	1423	1753	2137	2539
109	397	709	1049	1427	1759	2141	2543
113	401	719	1051	1429	1777	2143	2549
127	409	727	1061	1433	1783	2153	2551
131	419	733	1063	1439	1787	2161	2557
137	421	739	1069	1447	1789	2179	2579
139	431	743	1087	1451	1801	2203	2591
149	433	751	1091	1453	1811	2207	2593
151	439	757	1093	1459	1823	2213	2609
157	443	761	1097	1471	1831	2221	2617
163	449	769	1103	1481	1847	2237	2621
167	457	773	1109	1483	1861	2239	2633
173	461	787	1117	1487	1867	2243	2647
179	463	797	1123	1489	1871	2251	2657
181	467	809	1129	1493	1873	2267	2659
191	479	811	1151	1499	1877	2269	2663
193	487	821	1153	1511	1879	2273	2671
197	491	823	1163	1523	1889	2281	2677
199	499	827	1171	1531	1901	2287	2683
211	503	829	1181	1543	1907	2293	2687
223	509	839	1187	1549	1913	2297	2689
227	521	853	1193	1553	1931	2309	2693
							2699

For more prime numbers use MicroSMILE program *Numbers*.

The numbers
1, 2, 3, 4, 5, . . .
are called
natural numbers.

They are also known as *counting numbers* or *positive integers.*

The **natural numbers** together with **0** and **-1, -2, -3, -4, . . .** are called **integers.**

Rational numbers

Rational numbers are numbers which can be written in the form $\frac{p}{q}$ where **p** and **q** are integers with no common factors.

- All **integers** are rational numbers because they can be expressed in the form $\frac{p}{q}$.

  example

  $5 = \frac{5}{1}$ $\quad -4 = \frac{-4}{1}$ $\quad 272 = \frac{272}{1}$

- All **terminating decimals** are rational numbers because they can be expressed in the form $\frac{p}{q}$.

  example

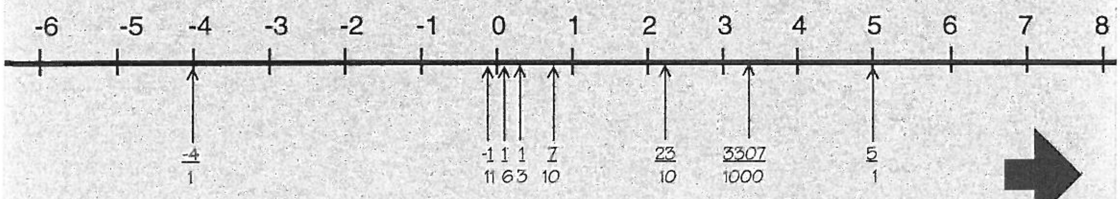
  $0.7 = \frac{7}{10}$ $\quad 2.3 = \frac{23}{10}$ $\quad 3.307 = \frac{3307}{1000}$

- All **recurring decimals** are rational numbers because they can be expressed in the form $\frac{p}{q}$.

  example

  $0.\dot{3} = \frac{1}{3}$ $\quad -0.\dot{0}\dot{9} = \frac{-1}{11}$ $\quad 0.1\dot{6} = \frac{1}{6}$

Because **rational numbers** can be expressed in the form $\frac{p}{q}$ they can be shown on a **number line**.



1. a) How many natural numbers are there?
   b) How many integers are there?
   c) How many rational numbers are there?

2. Give a rational number between:

   a) 2.51 and 2.52   b) $\frac{1}{4}$ and $\frac{1}{2}$ c) $\frac{4}{16}$ and $\frac{4}{15}$

3. Is it always possible to find a rational number between two other rational numbers?

   Explain your answer.

The following example shows a method to convert a recurring decimal into the form $\frac{p}{q}$.

Example To write $3.\dot{1}$ in the form $\frac{p}{q}$.

$3.\dot{1} = 3.111111...$ so $10 \times 3.\dot{1} = 31.111111111...$

and $1 \times 3.\dot{1} = 3.111111111...$

Subtracting $9 \times 3.\dot{1} = 28$

Therefore $3.\dot{1} = \frac{28}{9}$

4. Write these recurring decimals in the form $\frac{p}{q}$.

   a) $0.\dot{7}$ b) $1.\dot{3}\dot{4}$ (Hint: multiply by 100)

   c) $0.2\dot{6}$ d) $0.\dot{1}4285\dot{7}$ e) $0.0\dot{3}\dot{1}$

Irrational Numbers

Real numbers

The rational numbers together with the irrational numbers are the real numbers.

Rational numbers

. . . are numbers which can be written in the form $\frac{p}{q}$ where **p** and **q** are integers with no common factors. They include terminating and recurring decimals.

Irrational numbers

. . . are numbers which cannot be written in the form $\frac{p}{q}$ where **p** and **q** are integers with no common factors.

- Is **√3 rational** or **irrational**? Can it be written in the form $\frac{p}{q}$?
  Below is a proof that **√3** is irrational.

Proof by contradiction that √3 is irrational

Assume √3 is rational.

Assume the converse of what you are trying to prove.

$\sqrt{3} = \frac{p}{q}$ where p and q are integers with no common factors

$3 = \frac{p^2}{q^2}$

$3q^2 = p^2$

p^2 is a multiple of 3 ⇒ **p** is a multiple of 3

so $p = 3n$ where n is an integer

$3q^2 = (3n)^2$

$3q^2 = 9n^2$

$q^2 = 3n^2$

q^2 is a multiple of 3 ⇒ **q** is a multiple of 3

p and **q** are both multiples of 3 which contradicts the initial statement that **p** and **q** have no common factors.

This is a contradiction so the initial assumption is false.

√3 is irrational.

1. Prove by contradiction that **√2** is irrational.

√2 = 1.414213562. . .

√3 = 1.732050808. . .

π = 3.141592654. . .

These are all examples of non-terminating, non-recurring decimals.
They are **irrational numbers**.

Square roots of prime numbers are *always* **irrational.**

The square roots of some other numbers are also **irrational.**

Example: $\sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}$
5 is a prime number
$\sqrt{5}$ is a non-terminating, non-recurring decimal, it is **irrational,**
so $2\sqrt{5}$ is also **irrational.**

Not all square roots are **irrational.**

2. Which square roots are **rational** numbers?

3. For each number, say whether it is **rational** or **irrational.**

a) $\sqrt{8}$ b) $\sqrt{100}$ c) $\sqrt[3]{64}$ d) $\sqrt{\frac{9}{16}}$ e) $\sqrt{\frac{2}{25}}$

4. a) Is $(\sqrt{5})^2$ **rational** or **irrational**?
b) For which **irrational** numbers are their squares **rational**?

5. a) Explain why 2π is **irrational.**
b) Explain why $(2 + \pi)$ is **irrational.**
c) Explain why $(\pi + \sqrt{2})$ is **irrational.**

6. a) Explain why $(2 + \sqrt{2})$ is **irrational.**
b) Explain why $(2 + \sqrt{2})^2$ is **irrational.**

7. $(2 + \sqrt{3})$ and $(2 - \sqrt{3})$ are both **irrational.**
a) Is their sum **rational** or **irrational**?
b) Is their difference **rational** or **irrational**?
c) Is their product **rational** or **irrational**?

8. Give one pair of **irrational** numbers which give a **rational** number:
a) when multiplied.
b) when one is divided by the other.

9. Is x **rational** or **irrational** in each of the following equations?

a) $x^2 = 3$ b) $x^2 + 1 = 3$ c) $x^2 - 1 = 3$
d) $x^2 + x = 1$ e) $x^3 = 8$ f) $x^3 = 10$

PROOF BY CONTRADICTION

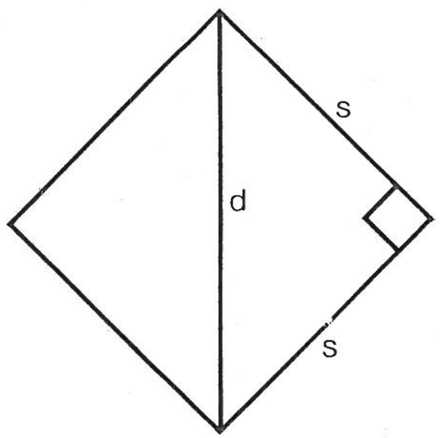
The ancient Greeks believed for a long time that any two lengths were "commensurable". By this they meant that the ratio of their lengths could be written as **p : q** where **p** and **q** were whole numbers. The whole foundation of Greek mathematics was shaken by the discovery, some time before 410 BC, that some ratios in geometrical figures were incommensurable. The proof given here is probably not the one that led to the initial discovery but it was well known to Aristotle. The proof is an example of proof by contradiction. The idea of such a proof is as follows:

(i) We assume the reverse of what we want to prove.

(ii) We try to determine the consequences.

(iii) If the consequences are nonsensical, then our original assumption must have been wrong.

We shall show that the ratio of the diagonal of a square to its side cannot be written in the form $\frac{p}{q}$ where **p** and **q** have no common factor.



$d^2 = ■^2 + ■^2$

$d^2 = 2■^2$

Turn over

1. By using Pythagoras' theorem, show that the ratio of the diagonal of a square to its side is $\sqrt{2}$

2. Assume $\sqrt{2} = \frac{p}{q}$, where **p** and **q** have no factors in common (apart, of course, from 1)

$$\frac{p^2}{q^2} = ■$$

$$\Rightarrow p^2 = ■$$

$\Rightarrow p^2$ is even *(why?)*

$\Rightarrow$ p is even *(why?)*

3. If **p** is even then **p** can be written as double another whole number so we could say $\mathbf{p} = \mathbf{2k} \Rightarrow \mathbf{p}^2 = ■$

4. Now write down an equation involving **q** and **k** but *not* **p**.
What does this tell you about **q**?
You should now have found that **p** and **q** are **both** even *(why?)*

5. Can you see why this leads to a contradiction?

Use the method of *'proof by contradiction'* to show that the planet Earth is not flat.

SMILE 0831

PRIMES & PROOF

Make sure you have: work-cards A, B, C
information card D
additional hints E

You don't have to work through them in any order and it doesn't really matter if you don't attempt all the work....

..... but you MUST work with a SMALL group of 2 or 3. You will need to talk about the work to understand it properly.