SQUARES & PRIMES

⋮

$29 = 5^2 + 2^2$

$31 =$

$37 = 6^2 + 1^2$

⋮

Some primes are the sum of 2 squares .....

Examine all the primes up to 100 and find whether they are the sum of 2 squares or not.

(1) What can you find out about the primes which are the sum of 2 squares? (Hint: multiples of 4)

(2) What do you notice about the pairs of square numbers which add up to prime numbers?

4n+1

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16
17	18	19	20
21	22	23	24

You should have noticed that:

a) If a prime number is the sum of 2 squares then it is one more than a multiple of 4 (it is of the form 4n + 1)

b) When the sum of 2 square numbers is prime, one of the squares is even and the other is odd.

Can you prove these statements? It is easier to take (b) first because this result will help to prove (a).

What sort of numbers are the sum of 2 squares?

$(\text{odd})^2 + (\text{odd})^2 = \text{odd} + \text{odd} =$

$(\text{even})^2 + (\text{even})^2 =$

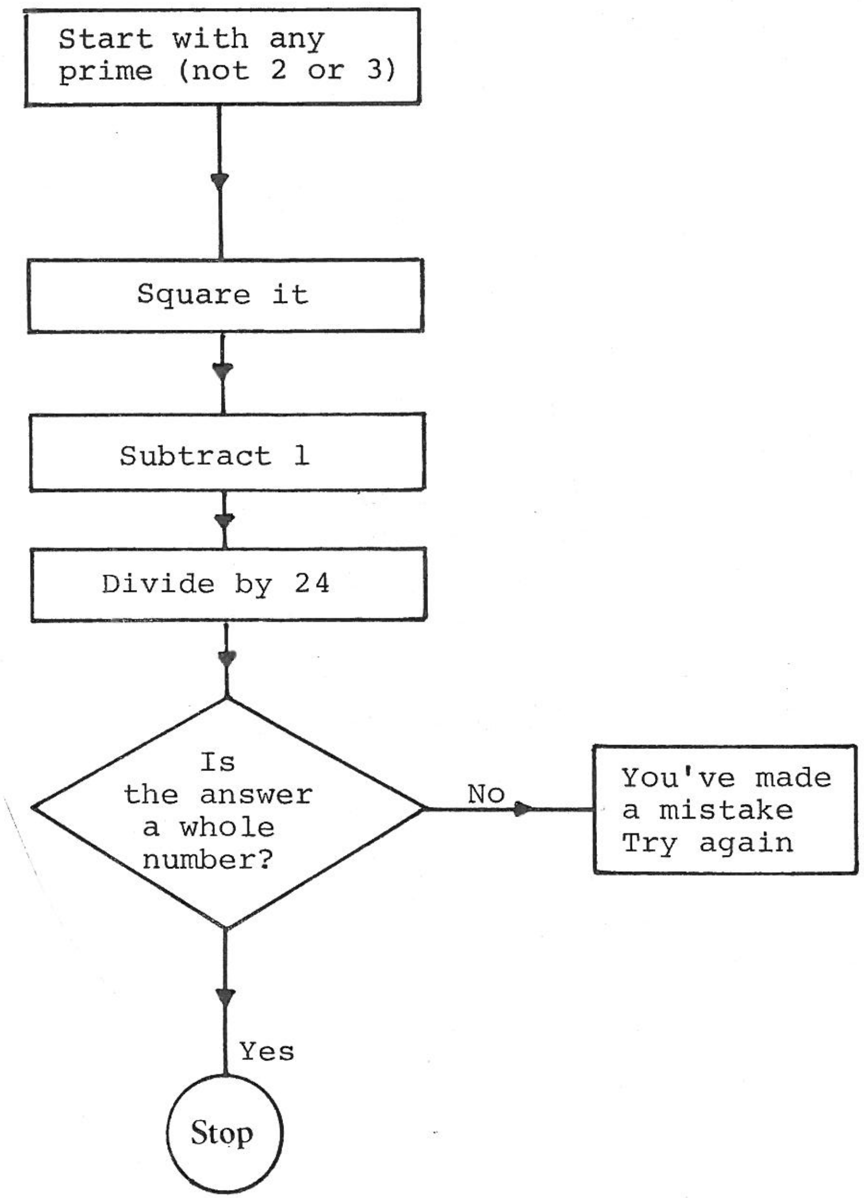
$(\text{even})^2 + (\text{odd})^2 =$

So if the sum of 2 squares is prime, ...

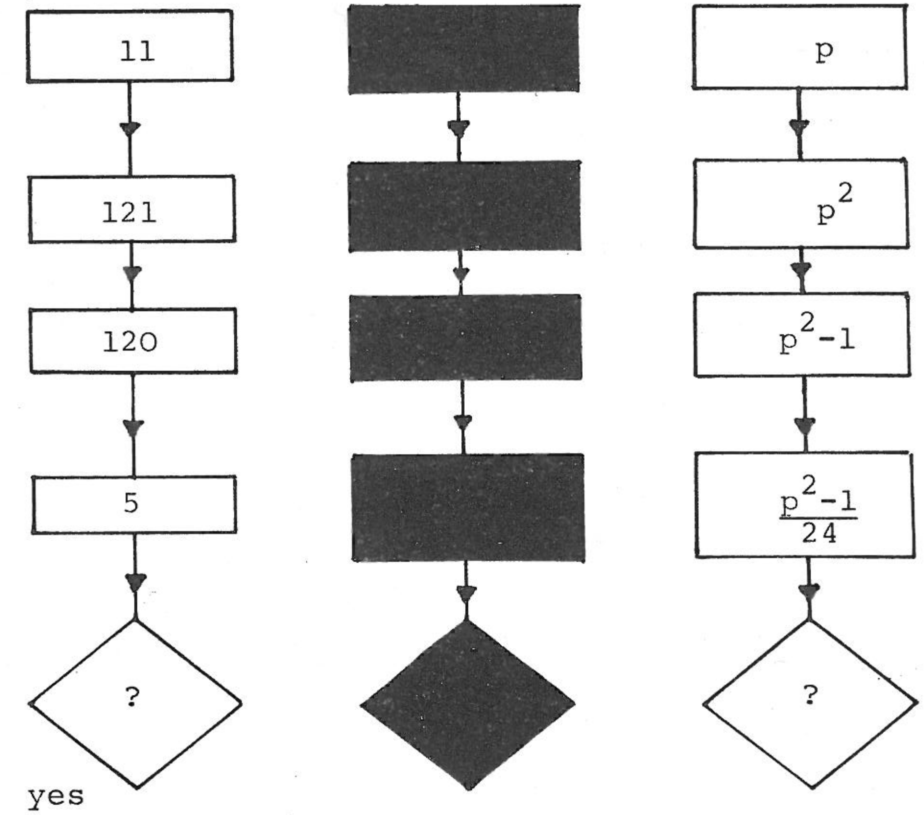
So we're only interested in $(\text{odd})^2 + (\text{even})^2$
An odd number is of the form (2p+1)
An even number is of the form 2q
So $(\text{odd})^2 + (\text{even})^2$ is of the form ....

SMILE 0831B

A PROOF ABOUT PRIMES



(1) Try 11: (2) Try some other primes (3) Try p:



If the final answer $\frac{p^2-1}{24}$ is always a whole number, what can you say about (p^2-1) and 24?

Turn over

(4) Is p^2-1 always a multiple of 24?

You can't test every prime (why not?)
but you might be able to find a proof.

Hints: (1) $p^2-1 = (p-1)(p+1)$

(2) 1,2,3,4,5, ..., (p-1),p,(p+1),...

(3) What factors do (p-1) and (p+1) have? Remember p is prime and so 2 and 3 cannot be factors of p.

If you need additional hints, look
at 0831E

Primes and Factorials

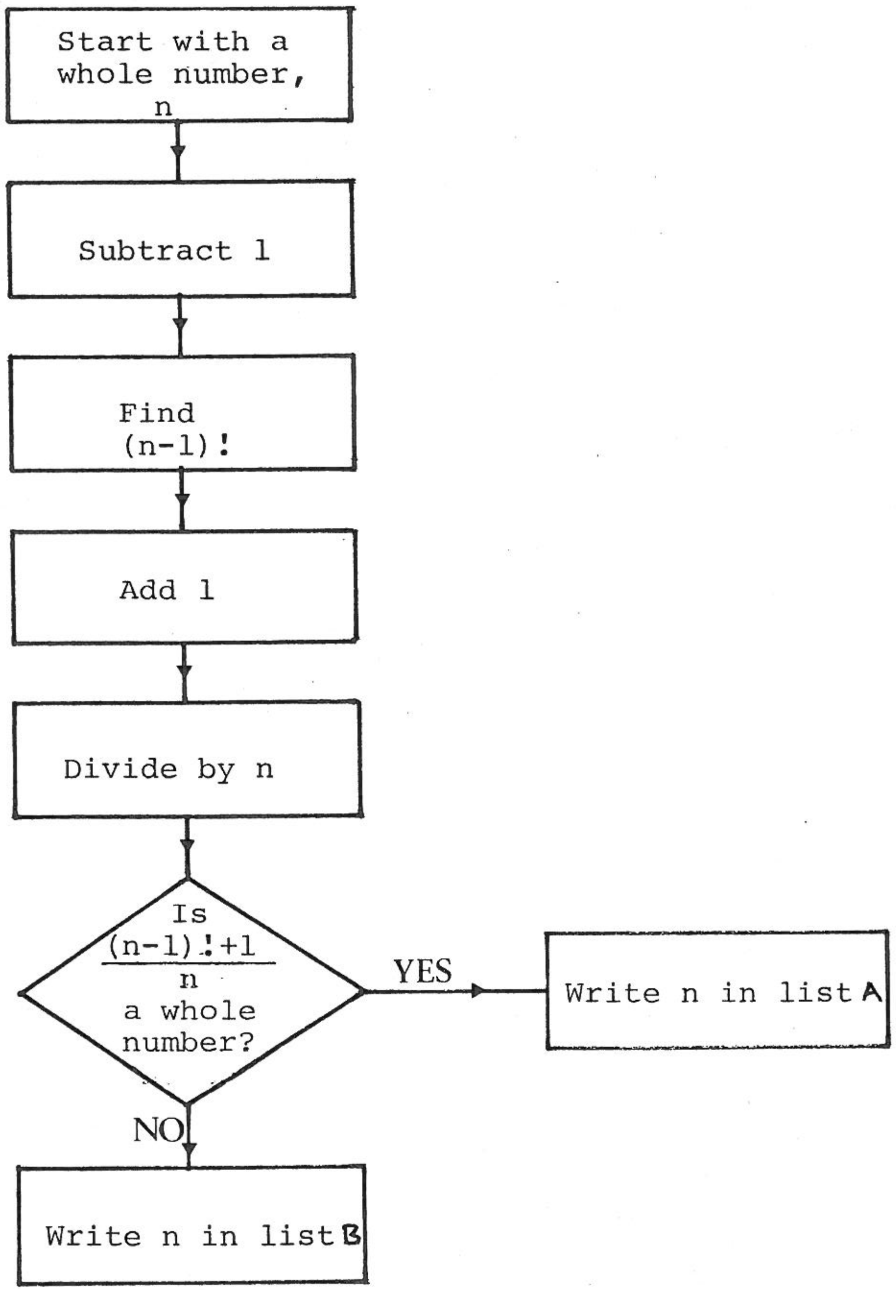
Factorial four means 4×3×2×1

4×3×2×1 is written 4!

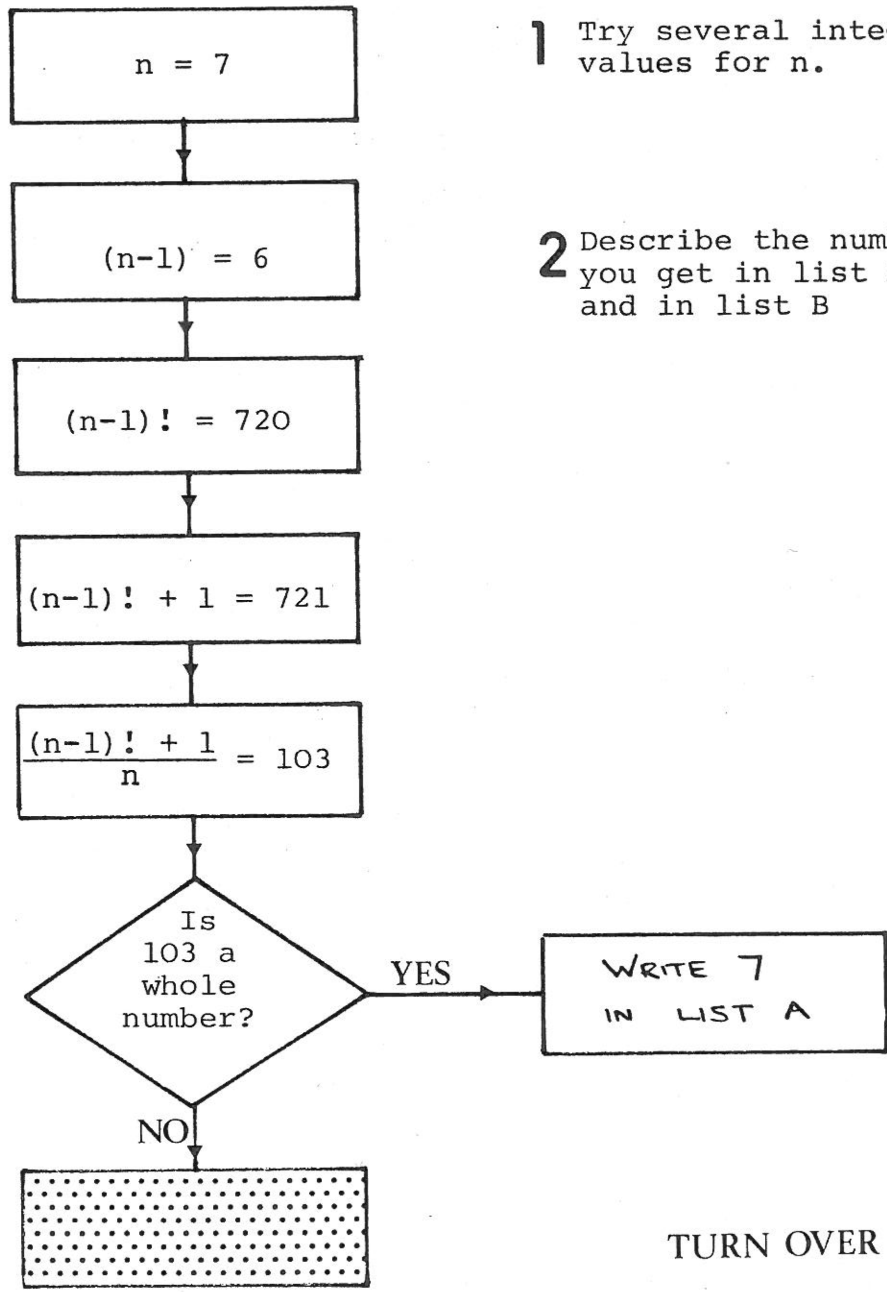
So 4! = 4×3×2×1 = 24

1! = 1
2! = 2
3! = 6
4! = 24
5! = 120
6! = 720
7! = 5 040
8! = 40 320
9! = 362 880
10! = 3 628 800
11! = 39 916 800
12! = 479 001 600
13! = 6 227 020 800
14! = 87 178 291 200
15! = 1 307 674 368 000
16! = 20 922 789 888 000
17! = 355 687 428 096 000
18! = 6 402 373 705 728 000
19! = 121 645 100 408 832 000
20! = 2 432 902 008 176 640 000

Check the first few numbers in this list, to make sure you understand it.



Try n = 7:



1 Try several intege
values for n.

2 Describe the numbe
you get in list A
and in list B

TURN OVER

You should have found that:

list A contains only [redacted] numbers

list B contains only [redacted] numbers

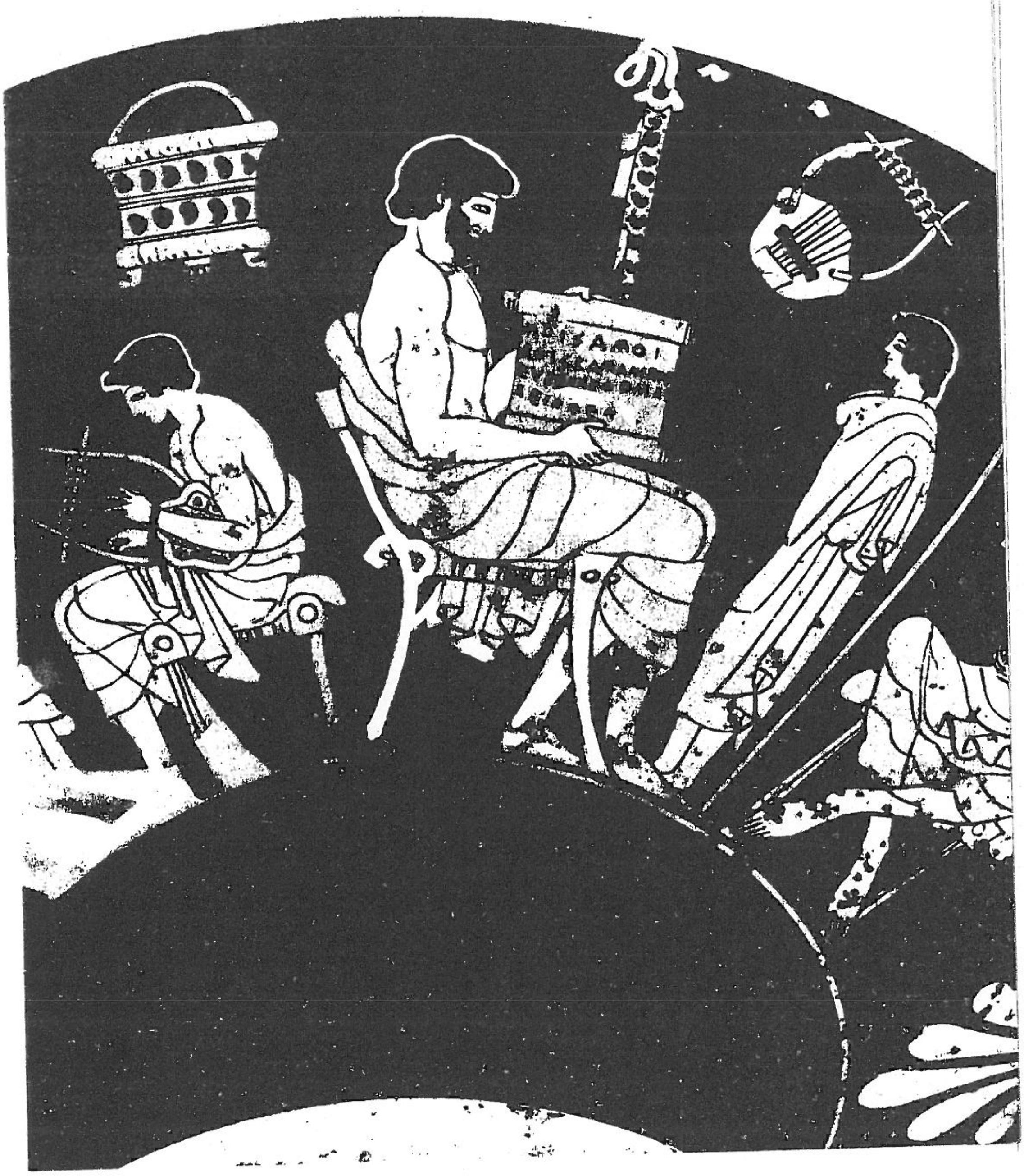
3 Can you see why <u>all</u> composite numbers <u>must</u> be in list B?

Hints: What are the factors of 8?
So why is 8 <u>not</u> a factor of
7x6x5x4x3x2x1 +1?

If you need additional hints look at 0831E.

Smile**0831D**

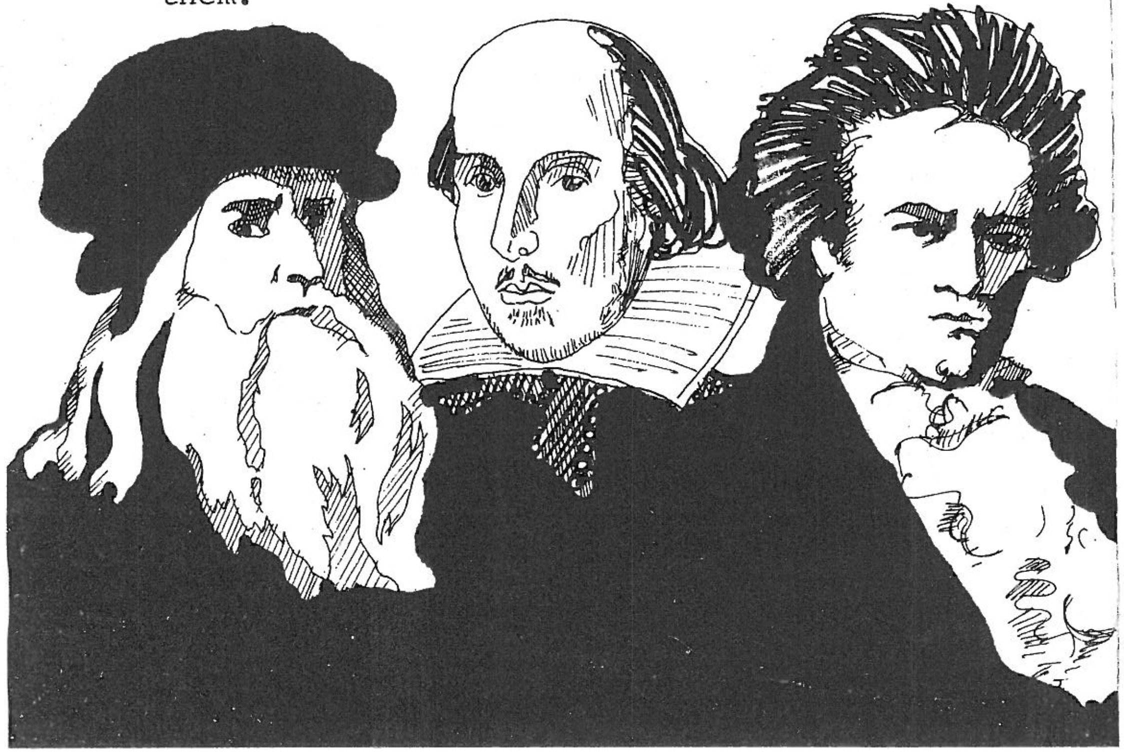
EUCLID



Shakespeare, Leonardo da Vinci and Beethoven all excelled in their particular subjects. We appreciate English literature by reading the works of Shakespeare; if we study art we look at Leonardo's paintings and how he worked; we learn more about music by studying the life and music of Beethoven.

For a fuller understanding of mathematics we study the work of great mathematicians. This booklet looks at two famous Greeks, Euclid and Pythagoras, who worked out principles so clearly that they laid a basis for the mathematics we use today. If you want to know about how these mathematicians lived and worked, how they and others developed mathematics, there is a list of interesting books on the back page.

When reading mathematics you should have a pencil and paper handy to make notes or work things out for yourself. You may have to read some parts twice before you fully understand them.



Euclid and Greek Mathemathics

Euclid was a Greek who lived around 300 BC - more than 2000 years ago. At this time, the centre of culture and learning had moved from Athens to the Egyptian port of Alexandria. The main attraction to scholars and teachers was the vast Alexandrian library of art, philosophy and science. The ideas and theories evolved over the previous centuries were collected together and Euclid not only taught the theories of older mathematicians like Pythagoras, but also wrote several books about Greek mathematics. Very few of these ancient books survive, but we do have copies of the works of Euclid, the most famous being his books about geometry called 'The Elements'. Hardly any of the ideas in The Elements were new, but their historical importance was in the way Euclid insisted on a proof for every point he described - thus laying a basis for mathematics for centuries to come. EUCLID REALISED THAT THERE ARE TWO STAGES IN MATHEMATICAL DEVELOPMENT: THE FIRST IS TO HAVE AN IDEA, MAKING AN HYPOTHESIS ABOUT MATHEMATICS WHICH WORKS IN A FEW CASES. THE SECOND STAGE IS TO PROVE THAT THIS GUESS OR HYPOTHESIS IS CORRECT, TO ESTABLISH THAT THE HYPOTHESIS WILL WORK IN ALL CASES.

The Infinity of Primes

An example of Euclid's approach is the theory of the infinity of prime numbers. Mathematicians before Euclid had suspected that there was an infinite number of primes - however large a prime number is, someone can always find a larger one. But Euclid actually proved that this is so, raising the guess to the level of a fact, established beyond doubt.

His method of proof is called "REDUCTIO AD ABSURDOM": assume the opposite of what you want to prove and then show that this leads to something absurb - a contradition.

Euclid's proof of the infinity of primes

(1) *Assume that the number of primes is <u>not</u> infinite.*

Suppose there are n different primes

$$p_1, \ p_2, \ p_3, \ \ldots\ldots\ldots\ldots, \ p_n$$

and the largest prime is p_n.

(2) *Multiply all n primes together and then add 1. Call the answer P.*

$$P = (p_1 \times p_2 \times p_3 \times \ldots\ldots \times p_n) + 1$$

(3) *P is not a multiple of any of the primes*

$p_1, p_2, \ldots\ldots p_n$ *because whichever one you divide by, the remainder is always one.*

So P must be prime.

(4) *The largest prime is* p_n *but P is clearly larger than this.*

(5) *.... so if there is a largest prime number* (p_n), *then there is an even larger prime (P)*

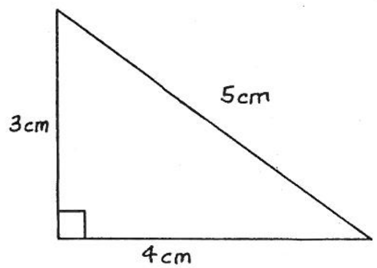
This is obviously nonsense and so the first assumption (1) must be wrong.

THE NUMBER OF PRIMES <u>IS</u> INFINITE

Any mathematician or computer could calculate 100 prime numbers. Any computer could calculate 1000 prime numbers. BUT IT TOOK THE GENIUS OF EUCLID TO ESTABLISH THAT HOWEVER MANY PRIME NUMBERS WERE CALCULATED IT WOULD BE POSSIBLE TO FIND MORE - THE NUMBER OF PRIMES IS INFINITE.

Pythagoras' Theorem is another example of the great power of mathematical proof. Mathematicians had known for many years before Pythagoras that the triangle shown has a right angle and also that:

$$3^2 + 4^2 = 5^2$$

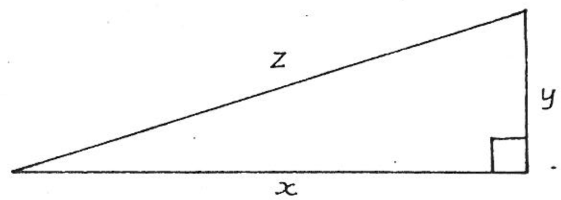


They knew that a similar rule worked for other right-angled triangles:

eg. $5^2 + 12^2 = 13^2$

But it was Pythagoras who PROVED that for every possible right-angled triangle the rule must work.

$$x^2 + y^2 = z^2$$



HOW DID PYTHAGORAS PROVE IT?

Consecutive Composites

Are there 5 consecutive numbers which are not prime? ....

are there 10 consecutive numbers which are not prime? ....

are there 50 consecutive numbers which are not prime? ....

Euclid proved that he could find <u>any number</u> of consecutive numbers that do not include a prime.

The proof is on the page opposite. Try to follow it through.

These 5 numbers are consecutive:

$6! + 2, \quad 6! + 3, \quad 6! + 4, \quad 6! + 5, \quad 6! + 6$

But 2 is a factor of 6!, so

2 is a factor of 6! + 2 (why?)

3 is a factor of 6! + 3 (why?)

4 is a factor of 6! + 4 (why?)

5 is a factor of 6! + 5 (why?)

6 is a factor of 6! + 6 (why?)

They all have factors; so none of them is prime.

Here are 10 consecutive numbers, none of which is prime:

$11! + 2, \quad 11! + 3, \quad 11! + 4, \quad 11! + 5, \quad 11! + 6,$

$11! + 7, \quad 11! + 8, \quad 11! + 9, \quad 11! + 10, \quad 11! + 11.$

Whatever number n is, none of the numbers in this list is prime.

$n! + 2, \quad n! + 3, \quad n! + 4, \quad \ldots\ldots\ldots \quad n! + n$

By making n as large as we like, we can make the list as long as we like.

Do you understand this proof?

Write down 20 consecutive numbers which are not prime.

Explain why none of them is prime.

If you want to find out more about Euclid, Pythagoras and Greek mathematics, the following books are interesting:

1. Life Science Library: Mathematics-D.Bergamini

2. Mathematics, A Human Endeavour-H.Jacobs
(W.H.Freeman & Co.)

3. Greek Mathematicians (Exploring Maths.) French
(McGraw Hill)

4. Mathematics and Imagination-Kasner & Newman
(Bell)

5. A Short Account of the History of Mathematics
- W.W. Rouse Ball (Dover)

6. A History of Mathematics C Boyer
(Wiley International)

Additional Hints

The work on the back pages of 0831B and 0831C is very difficult. Use this leaflet sensibly for hints if you need them. Look at hint 1 and try again before you look at hint 2, and so on.

Turn to the index on page 2 to find the hint you need.

Index:

A Proof About Primes (0831B)

Hint 1

p must be an odd number, so (p-1) and (p+1) must both be ....

Every other even number is also a multiple of ......

so either (p-1) or (p+1) must be ....

2
(4)
6
(8)
10
(12)
14
.
.
.

Primes and Factorials (0831C)

Hint 1

Why is 8 not a factor of 7! + 1 ?

..... *4 is not a factor of 7! + 1 (why not?), and if 8 were a factor 4 would be a factor too.*

..... *similarly,*

2 is not a factor of
7! + 1
and if ....

A Proof About Primes (0831B)

Hint 2

(p-1), p, (p+1) are consecutive numbers, so one of them must be a multiple of 3

p is not a multiple of 3, so ......

Primes and Factorials (0831C)

Hint 2

Why is 4 not a factor of 7! ?

7! = 7x6x5x4x3x2x1

So 4 is a factor of 7!

So ......

A Proof About Primes (0831B)

Hint 3

Either (p-1) or (p+1) is a multiple of 3 and

either (p-1) or (p+1) is a multiple of 4 ....
..... *and whichever is not a multiple of 4 is still a multiple of 2*

So (p-1) (p+1) must be .....