SMILE WORKCARDS

Translations and Vectors Pack Two

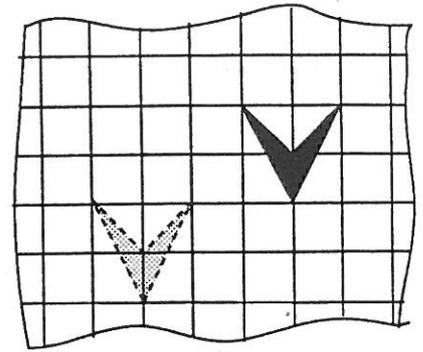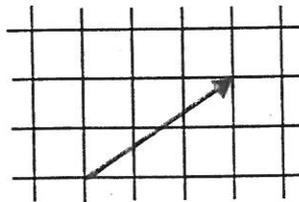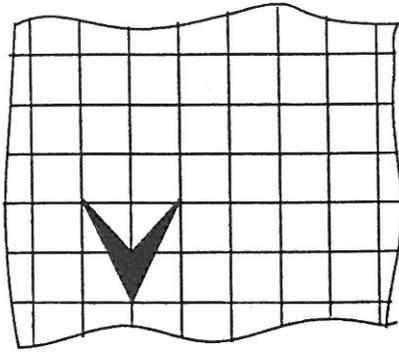## Contents

TRANSLATIONS

You may need tracing paper.

A translation moves *every* part of a shape
- the same direction
- the same distance

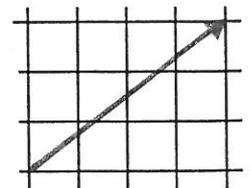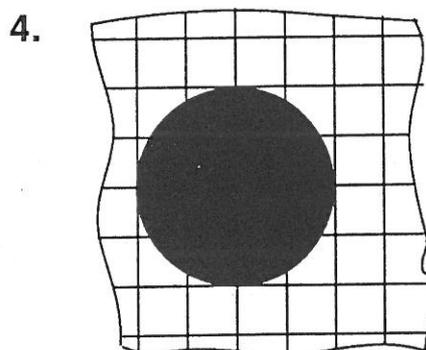This shape . . .   . . . is moved by
                         this translation . . .   . . . to this new position.

Show these translations on squared paper.

1.

2.

3.

4.
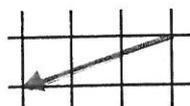
Combined Translations
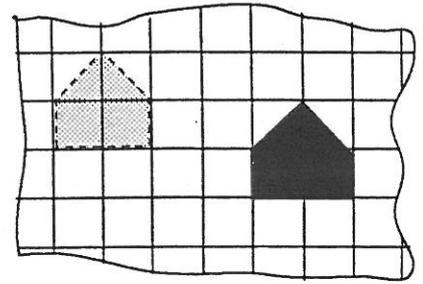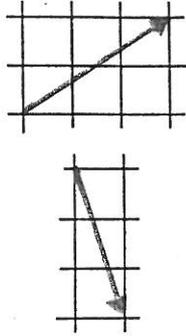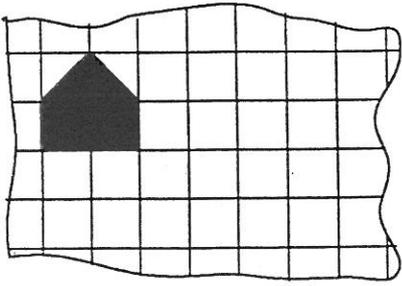
This shape . . .   . . . is moved by
                            these two translations . . .   . . . to this position.

5.   Show the following translation on squared paper.
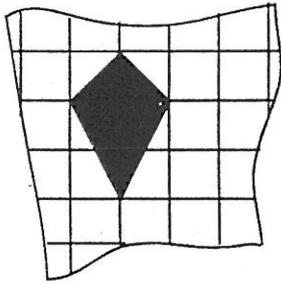
Find a single translation
to replace these two
translations.

Turn over

Find a single translation to replace those in each of **6** and **7**.

6.
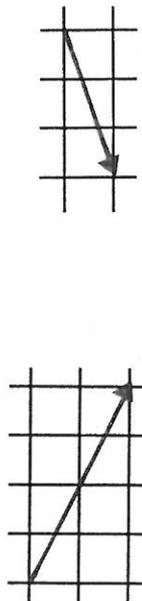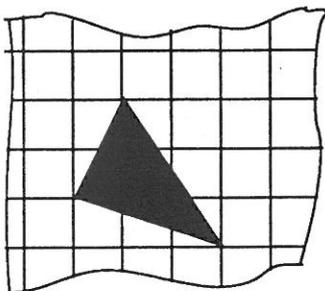
7.
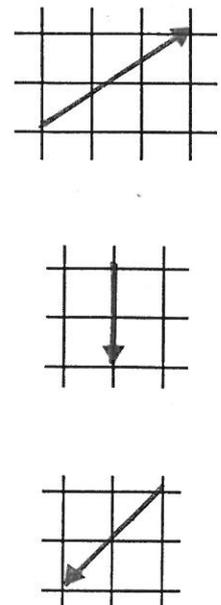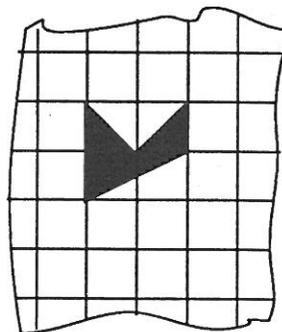
Journeys

The diagram shows a journey in two stages.

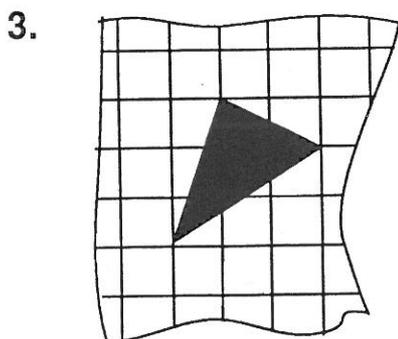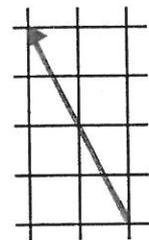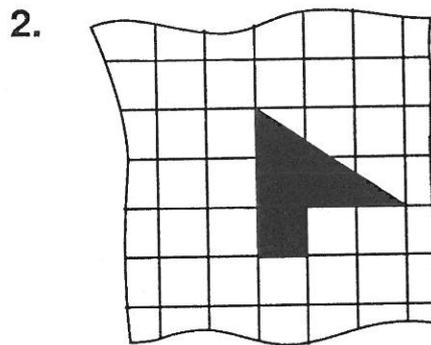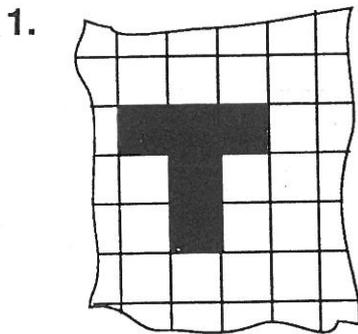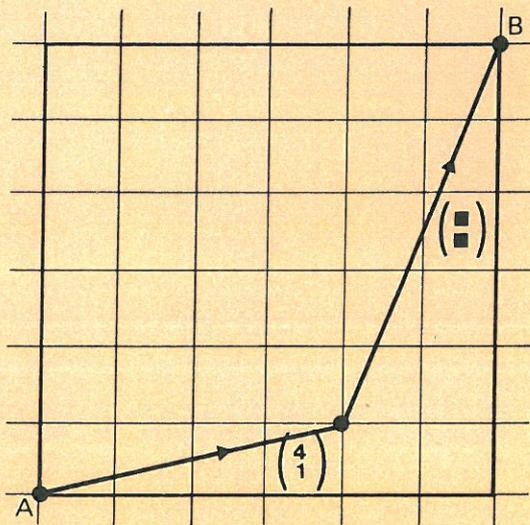The vector which describes the first stage is $\binom{4}{1}$.

1) Copy the diagram and label both vectors.

2) On the same grid draw 3 more two-stage journeys which start at A and finish at B.

3) Copy and complete the table.

4) Look carefully at the numbers in the vectors and write down the connection between them.

Journey A to B	
Direct Vector	**Two Stage Journey**
$\binom{6}{6}$	$\binom{4}{1}$, $\binom{\blacksquare}{\blacksquare}$
$\binom{6}{6}$	$\binom{\blacksquare}{\blacksquare}$, $\binom{\blacksquare}{\blacksquare}$
$\binom{6}{6}$	$\binom{\blacksquare}{\blacksquare}$, $\binom{\blacksquare}{\blacksquare}$
$\binom{6}{6}$	$\binom{\blacksquare}{\blacksquare}$, $\binom{\blacksquare}{\blacksquare}$
$\binom{6}{6}$	$\binom{\blacksquare}{\blacksquare}$, $\binom{\blacksquare}{\blacksquare}$
$\binom{6}{6}$	$\binom{\blacksquare}{\blacksquare}$, $\binom{\blacksquare}{\blacksquare}$
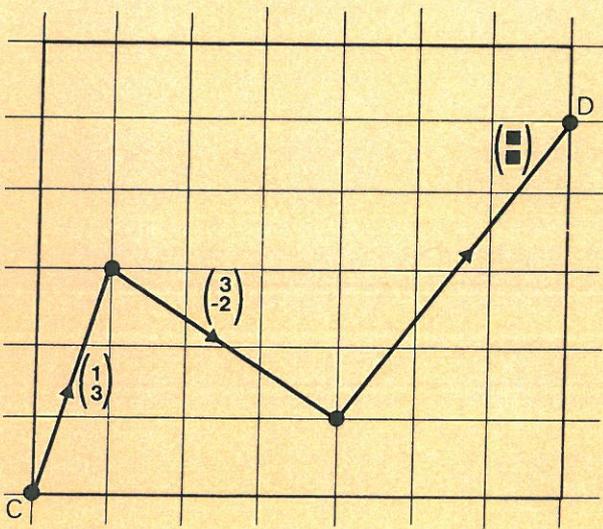
This is a three-stage journey — from C to D.

5) Explain why the vector for the middle stage is $\binom{3}{-2}$ and not $\binom{3}{2}$.

   (If you are unsure see Smile card 1309 — More Vector Messages).

6) Use vectors to describe some different three-stage journeys from C to D. Record the results in a table as you did for question (3).

7) Write down the connections between the numbers in the vectors.
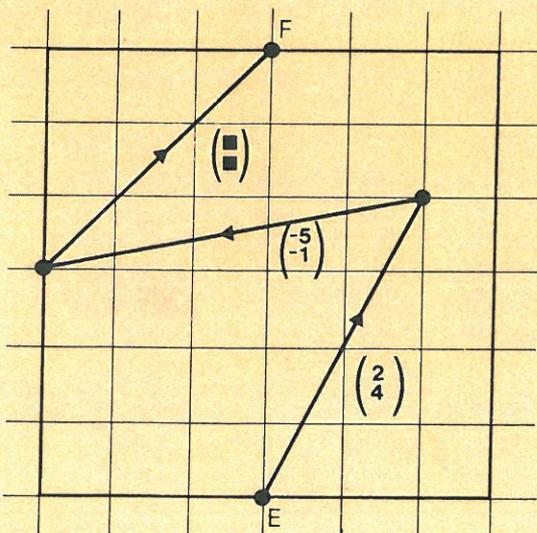
8) Record the vectors for some different journeys (two-stage, three-stage, . . .) from E to F.

9) What is the connection between the vectors you recorded in (8) and the vector $\binom{0}{6}$?

10) Use the direct vector to check that your work is correct.

11) Choose another pair of points, G and H, and repeat.

Vectors and Squares

ABCD is a square.
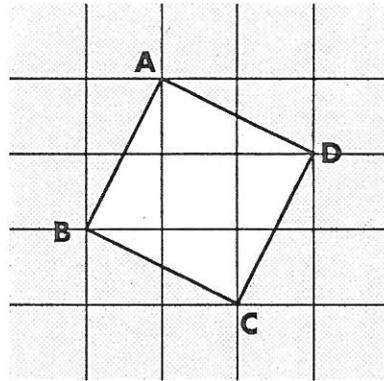
It can be described by these vector moves.

$$\vec{AB} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \quad \vec{BC} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \vec{CD} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \vec{DA} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

This is another square.
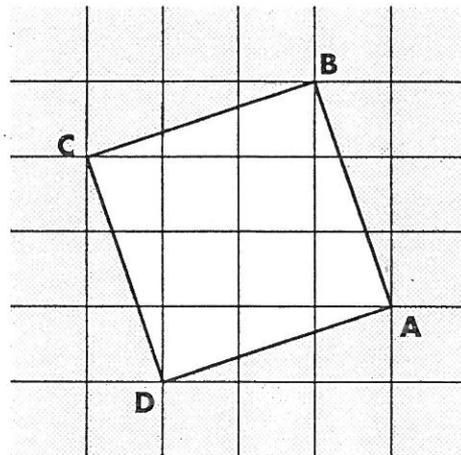
● Describe it by vector moves.

● Draw other squares and describe them by vector moves.

Making a generalisation

● If the vector move $\vec{AB}$ is $\begin{pmatrix} x \\ y \end{pmatrix}$ what are the other vector moves of the square in terms of **x** and **y**?

Testing the generalisation

● Use your generalisation to predict the other 3 vectors when $\vec{AB}$ is $\begin{pmatrix} -5 \\ 2 \end{pmatrix}$.

Check that these vectors make a square.
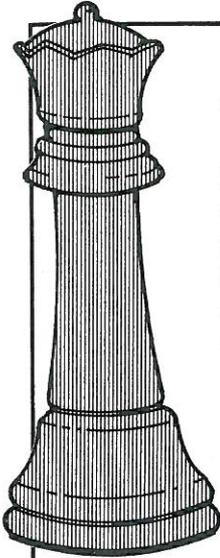
Avoiding each other

In chess a queen can move any distance in a straight line: along rows, along columns, or along diagonals. Normally each player only has one queen . . .
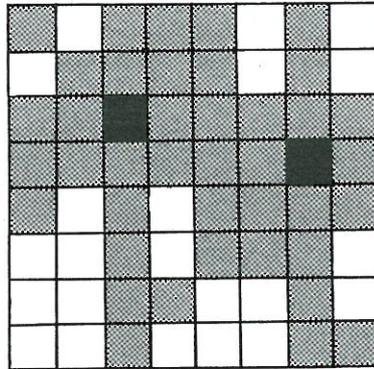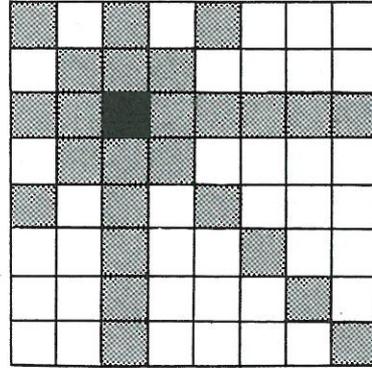
. . . but if you could have more than one queen, you could protect many more squares.

Problem: *Can you arrange 8 queens on a chess-board (8x8) so that no two queens are in line with each other?*

This problem is quite hard. It will be helpful to use the micro program called QUEENS. This program enables you to draw several patterns very quickly. You can also rub out queens you don't want and so it is easier to explore different arrangements.

With the micro program you can work with boards of different sizes, so there are many more problems you can try . . .

FORCE MEET

You will need a large sheet of squared paper and the 9 Force Meet playing cards.

Force Meet is a game for two players. One player follows a rule — the opponent does not know what this is. The object of the game is for the opponent to guess the rule and use it to force the other player to converge on his position.

To play

Shuffle the cards. Place them face down in a pile on the table.

Each player marks a starting position on the grid.

Lead player draws a straight line move of any length.

Second player chooses a card. This gives the rule for the whole game. Second player follows this rule and moves.

Lead player moves again.
Second player follows the same rule.
......and so on......
Lead player wins if he can force the second player to meet him.

Take turns to lead.

FORCE MEET

Move **twice the distance** as your opponent...... in the **same direction**
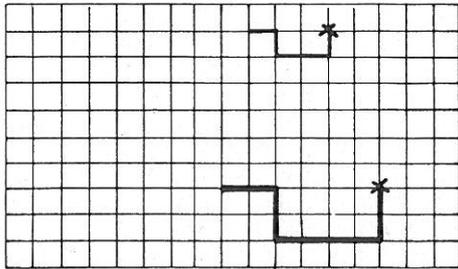
Any rule you choose.

Move **half the distance** as your opponent...... ......in the **opposite direction**.
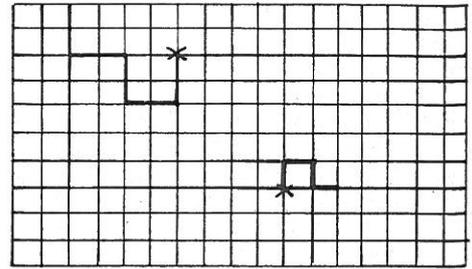
Move the **same distance** as your opponent...... in the **opposite direction**
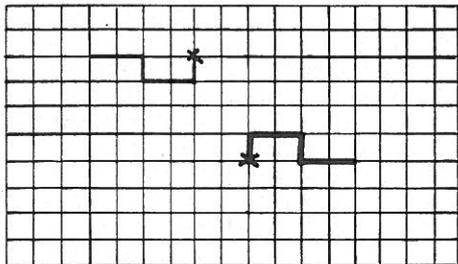
Any rule you choose.

Move the **same distance** as your opponent...... ......always a **90° anti-clockwise turn** from his direction.
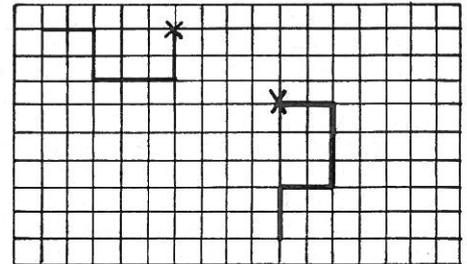
Move **three times** your opponent's distance.... always a **90° clockwise turn** from his direction.
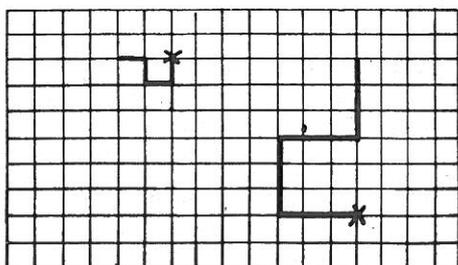
Any rule you choose.

If opponent moves **north** or **south**: move same distance in opposite direction.......

if opponent moves **east** or **west**: move same distance in same direction.

Smile 1013

Vector Magnitude

A **vector** is defined by **magnitude** (length) and **direction**.

A $\vec{PQ} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

The triangle PQN has a right angle at N.

1. What is the length PN?
2. What is the length QN?
3. Use Pythagoras' theorem to calculate the magnitude (length) of $\vec{PQ}$.
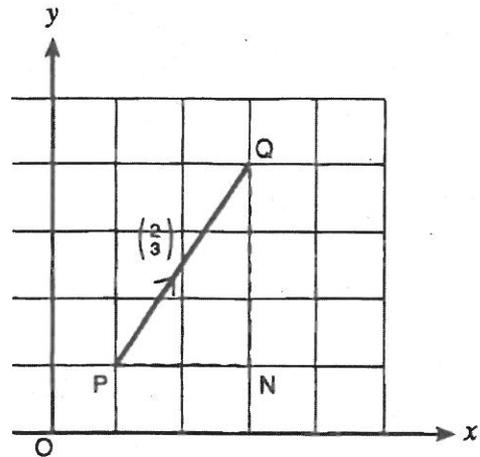
B Find the magnitude of the vectors, you may like to draw diagrams:

1. $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ 2. $\begin{pmatrix} 5 \\ 12 \end{pmatrix}$ 3. $\begin{pmatrix} 2 \\ 7 \end{pmatrix}$

C Find the magnitude of these vectors:

1. $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$ 2. $\begin{pmatrix} 3 \\ -3 \end{pmatrix}$ 3. $\begin{pmatrix} -2 \\ -8 \end{pmatrix}$ 4. $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$

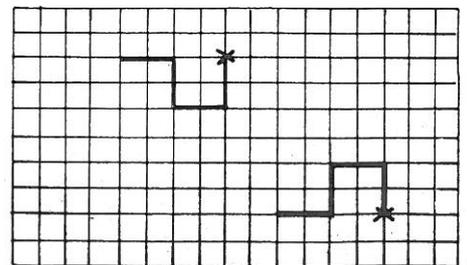**D** Find the magnitude of these vectors:

1. $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$ 2. $\begin{pmatrix} a_1 \\ 0 \end{pmatrix}$ 3. $\begin{pmatrix} 0 \\ a_2 \end{pmatrix}$ 4. $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

E In this diagram, the vector $\vec{PQ} = a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

Write down the length of:

1. PN 2. QN

Find the magnitude of $\vec{PQ}$ in terms of a_1 and a_2.
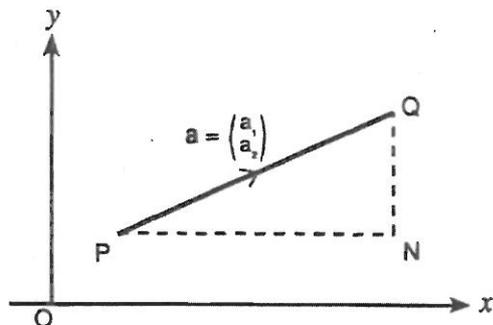
F $\begin{pmatrix} a \\ 0 \end{pmatrix}$, $\begin{pmatrix} 3 \\ b \end{pmatrix}$, $\begin{pmatrix} c \\ -4 \end{pmatrix}$, $\begin{pmatrix} -5 \\ d \end{pmatrix}$ are vectors of length 5.

Find a, b, c and d.

Turn over.

G The vector $\overrightarrow{PQ} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$ is a vector in three dimensions.

1. Calculate the length PR.
2. Use the right-angled triangle PRQ to calculate the magnitude of the vector $\overrightarrow{PQ}$.
3. Calculate the length QN.
4. Is QNP a right-angled triangle?
5. Calculate the magnitude of $\overrightarrow{PQ}$, using your value for QN.
   Check that both your calculations for the magnitude of $\overrightarrow{PQ}$ agree.
6. Is it true that $PQ^2 = 3^2 + 2^2 + 6^2$?
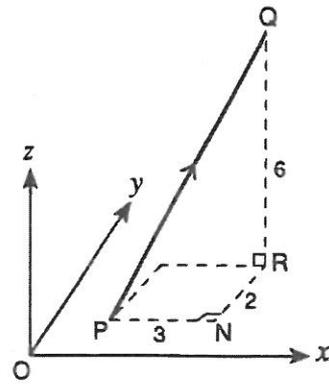
H The vector $\overrightarrow{OR}$ is the diagonal of a box. OP, OT, and OV lie along the x, y and z axes.
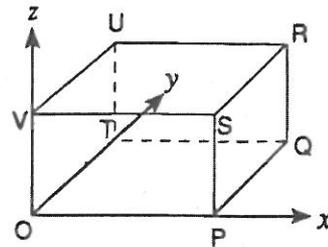
If $\overrightarrow{OR} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, what is the length of: 1. OV   2. OP   3. OT   4. PS   5. OS

Use the fact that $OR^2 = OS^2 + SR^2$ in the right-angled triangle OSR to express OR^2 in terms of a_1, a_2 and a_3.

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, then the length of $\mathbf{a} = \sqrt{(a_1^2 + a_2^2)}$.

If $\mathbf{a}$ is in three dimensions and $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ then the length $\mathbf{a} = \sqrt{(a_1^2 + a_2^2 + a_3^2)}$.

J Find: 1. $(PQ)^2$ if $\overrightarrow{PQ} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ 2. $(HK)^2$ if $\overrightarrow{HK} = \begin{pmatrix} -1 \\ 2 \\ 12 \end{pmatrix}$ 3. The length of PQ and HK.

K Calculate the magnitude of the vectors:
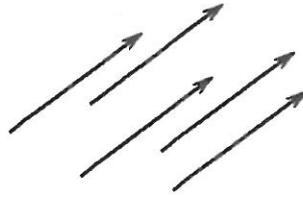
1. $\begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$ 2. $\begin{pmatrix} -1 \\ 2 \\ 6 \end{pmatrix}$ 3. $\begin{pmatrix} 4 \\ -4 \\ 4 \end{pmatrix}$
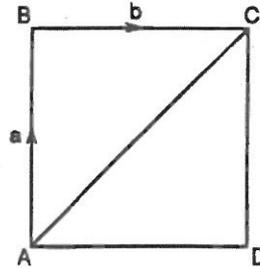
Vectors

A vector is defined by **magnitude** (length) and **direction**.

All of these vectors are **equivalent**.
They have the same magnitude and
direction.

In this square: the vector $\overrightarrow{AB}$ can be written as **a**.

the vector $\overrightarrow{BC}$ can be written as **b**.

The vectors **a** and **b** are equal in magnitude,
but they have different directions, so they are
not equivalent.

The vector $\overrightarrow{DC}$ is equivalent to $\overrightarrow{AB}$, so $\overrightarrow{DC}$ = **a**.

The vector $\overrightarrow{CD}$ is equivalent to $\overrightarrow{AB}$, but in the opposite direction, so $\overrightarrow{CD}$ = -**a**.

- What other vector is equivalent to **b**?

The **resultant** of the two vectors **a** and **b** is the direct journey which is
equivalent to **a** followed by **b**.

In the square above, the resultant of **a** followed by **b** is $\overrightarrow{AC}$, $\overrightarrow{AC}$ = **a** + **b**.

$\overrightarrow{AC}$ in terms of **a** and **b** is **a** + **b**.

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1. KLMN is a parallelogram.
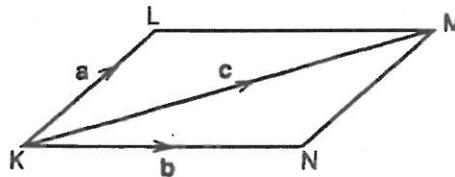
(i) Vector $\overrightarrow{KL}$ = **a**   Which other vector is equal to **a**?
    Vector $\overrightarrow{KN}$ = **b**   Which other vector is equal to **b**?

(ii) Which sum of two vectors is equivalent to **c**?

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2. ABCD is a parallelogram.
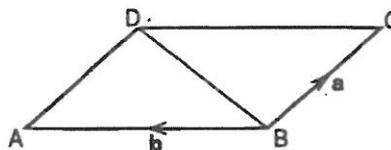
What is vector $\overrightarrow{BD}$ in terms of **a** and **b**?

Turn over

3. L is the mid point of KM.
$\vec{KL} = \mathbf{a}$.
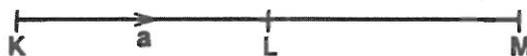
Give each of the vectors $\vec{LM}$ and $\vec{KM}$ in terms of **a**.

K ——— a ——— L ——— M

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4. R is the mid point of PQ.
$\vec{PQ} = \mathbf{b}$.

Give $\vec{PR}$ and $\vec{RQ}$ in terms of **b**.

P ——— R ——— Q

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5. Give (i) $\vec{TV}$ in terms of **a** and **b**.

(ii) $\vec{VT}$ in terms of **a** and **b**.
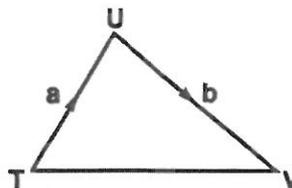
In this diagram, $\vec{PQ} = \mathbf{a}$, $\vec{RQ} = \mathbf{b}$ and $\vec{PR} = \mathbf{c}$

$\mathbf{c} + \mathbf{b} = \mathbf{a}$
$\mathbf{c} = \mathbf{a} - \mathbf{b}$
$\mathbf{c} = \mathbf{a} + (-\mathbf{b})$
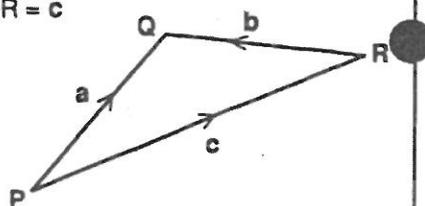
In each of these find **c** in terms of **a** and **b**.

1.

2.

3.

4.
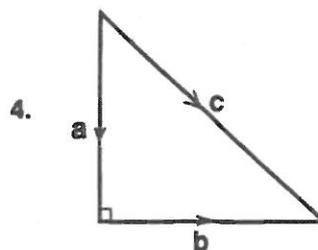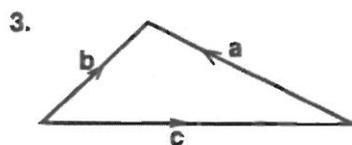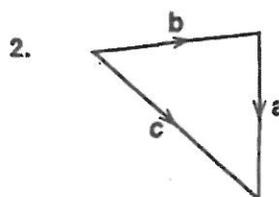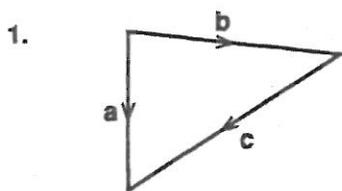
5. ABC is an isosceles triangle

AB = AC.

AD is perpendicular to BC.

Give i) $\vec{BC}$ in terms of **a** and **b**.
ii) $\vec{BD}$ in terms of **a** and **b**.

MORE VECTORS

In the diagram P is the mid-point of XY and
Q is the mid-point of XZ

XP = PY

XQ = QZ

$\overrightarrow{YP}$ = **a**

$\overrightarrow{QZ}$ = **b**

$\overrightarrow{PX}$ = **a**

$\overrightarrow{XQ}$ = **b**

$\overrightarrow{PQ}$ in terms of **a** and **b** is **a** + **b**

$\overrightarrow{YZ}$ in terms of **a** and **b** is 2**a** + 2**b**

This shows that $\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{YZ}$.

a   OAB is a triangle where $\overrightarrow{OA}$ = **a**, $\overrightarrow{OB}$ = **b** and $\overrightarrow{AP}$ = $\overrightarrow{PB}$ = **c**.

● Use triangle OAP to find $\overrightarrow{OP}$ in terms of **a** and **c**.

● Use triangle OPB to find $\overrightarrow{OP}$ in terms of **b** and **c**.
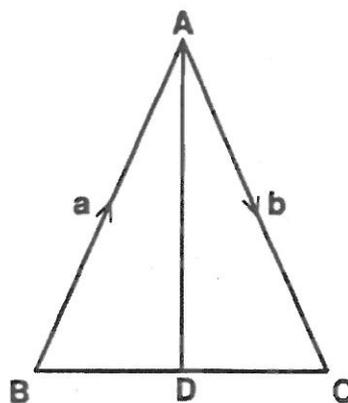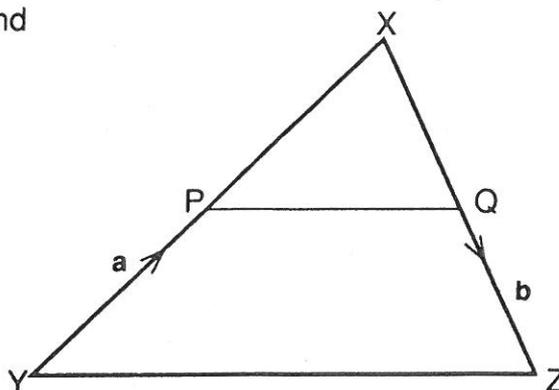
● Show that $\overrightarrow{OP} = \frac{1}{2}$ (**a** + **b**).
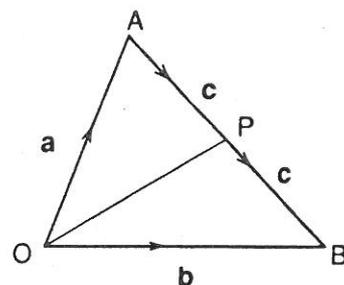
b   In triangle XYZ, P, Q, R are the mid points of YZ, XZ, XY respectively.
Vectors $\overrightarrow{YZ}$ = **x**, $\overrightarrow{ZX}$ = **y**, $\overrightarrow{XY}$ = **z**.

● Find $\overrightarrow{XP}$ in terms of **y** and **z**.

● Find $\overrightarrow{ZR}$ in terms of **x** and **y**.

● Find $\overrightarrow{YQ}$ in terms of **x** and **z**.

● Use this to find the value of $\overrightarrow{XP} + \overrightarrow{ZR} + \overrightarrow{YQ}$.
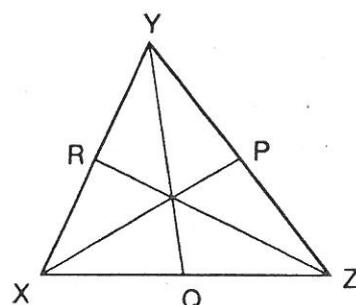
c   OAB is a triangle where $\overrightarrow{OA}$ = **a**, $\overrightarrow{OB}$ = **b**, $\overrightarrow{AW}$ = 2**d** and $\overrightarrow{WB}$ = **d**.

● Find $\overrightarrow{OW}$ in terms of **a** and **b**.
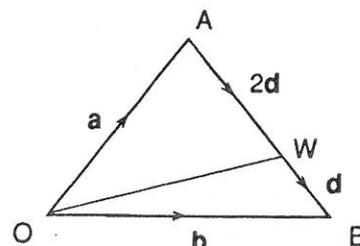
Column Vectors
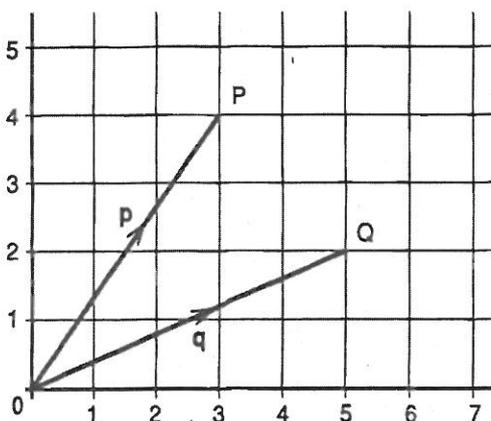
$$\overrightarrow{OP} = p = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \qquad \overrightarrow{OQ} = q = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is the column vector for **p**

$\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ is the column vector for **q**

To find the column vector for $\overrightarrow{PQ}$, write the vector equation

$$
\begin{aligned}
\overrightarrow{PQ} &= \overrightarrow{PO} + \overrightarrow{OQ} \\
&= \text{-}p + q \\
&= -\begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix} \\
&= \begin{pmatrix} -3 \\ -4 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix} \\
&= \begin{pmatrix} 2 \\ -2 \end{pmatrix}
\end{aligned}
$$

Check $\overrightarrow{PQ} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ from the diagram.

1. $\overrightarrow{OA} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$

$\overrightarrow{OB} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

(a) (i) Find the column vector for $\overrightarrow{BA}$.

M is the mid point of the line BA.

(ii) Find the column vector for $\overrightarrow{BM}$.

(b) Find the co-ordinates of M.

(c) Calculate the length of OM.

(d) What is the gradient of OM?

2. (a) $\begin{pmatrix} 4 \\ 6 \end{pmatrix}$ is column vector **a**.
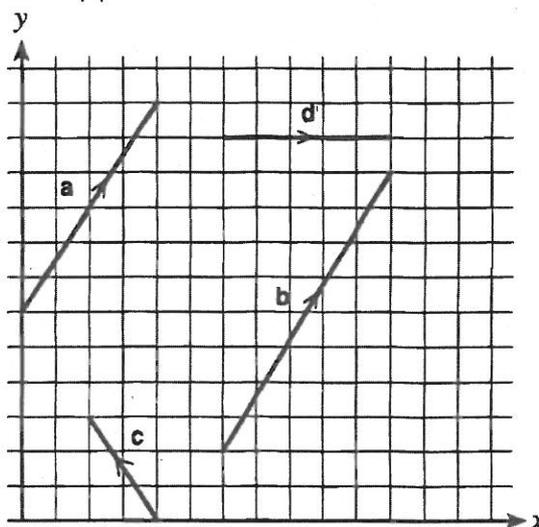
Express as column vectors

(i) **b** (ii) **c** (iii) **d** (iv) **a** + **c**.

(b) The vectors **m**, **n** and **s** are shown on the diagram.
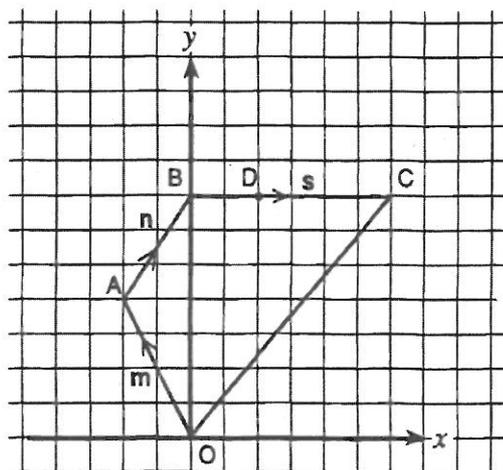
Write the following vectors in terms of **m**, **n**, and **s**.

(i) $\overrightarrow{OB}$ (ii) $\overrightarrow{AC}$

(iii) $\overrightarrow{AD}$, where $\overrightarrow{BD} = \frac{1}{3} \overrightarrow{BC}$

(c) Calculate the angle which the vector $\overrightarrow{OD}$ makes with the x-axis.

DIVIDING IN A GIVEN RATIO

In this diagram, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

The ratio of $AP : PB = 1 : 4$.

The column vector for $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

The column vector for $\mathbf{b} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$

Vector $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

Vector $\overrightarrow{AP}$ is $\frac{1}{5}$ of AB $= \frac{1}{5}(\mathbf{b} - \mathbf{a})$
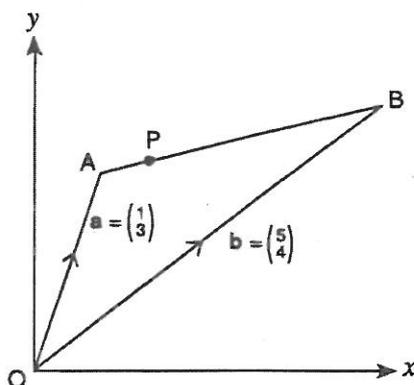
Similarly $\overrightarrow{PB} = \frac{4}{5}(\mathbf{b} - \mathbf{a})$

A This diagram shows vectors $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

The ratio of $AP : PB = 3 : 2$.

Express in terms of $\mathbf{a}$ and $\mathbf{b}$:

1. $\overrightarrow{AB}$ 2. $\overrightarrow{AP}$ 3. $\overrightarrow{PB}$

B 1. Complete the vector equation

$\overrightarrow{OP} = \overrightarrow{OA} + \boxed{?}$

2. Vector $\overrightarrow{OP} = \mathbf{r}$. Use this to express $\mathbf{r}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ and simplify your result.
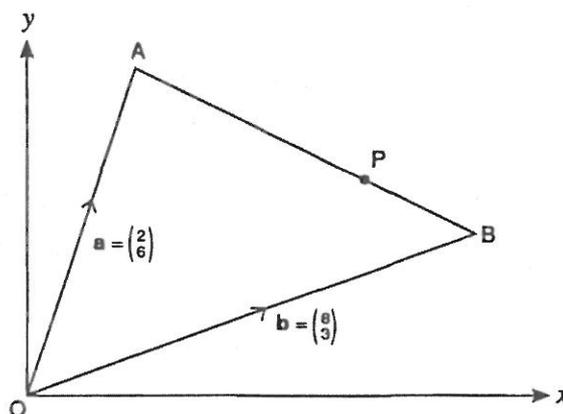
C Substitute the column vector values of $\mathbf{a}$ and $\mathbf{b}$ to find $\mathbf{r}$ as a column vector.

What are the co-ordinates of P?

D Q lies on AB so that $AQ : QB = 3 : 7$.

$\overrightarrow{OQ} = \mathbf{s}$.

1. Show that $\mathbf{s} = \frac{1}{10}(7\mathbf{a} + 3\mathbf{b})$.

2. Use the column vector values of $\mathbf{a}$ and $\mathbf{b}$ above to find the co-ordinates of Q.

E Look at your results of $\mathbf{r}$ and $\mathbf{s}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

1. Use the pattern in these results to write down $\overrightarrow{OT} = \mathbf{t}$ in terms of $\mathbf{a}$ and $\mathbf{b}$, where T lies on AB and $AT : TB = 1 : 2$.
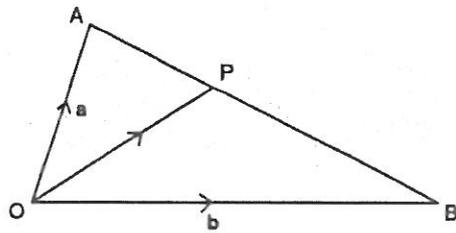
2. What are the co-ordinates of T?

3. Plot the points of A, B and T. Measure to check that $2AT = TB$.

Turn over.

F To prove these results for any points, let O, A, and B be *any* three points and $\overrightarrow{OA}$ = **a**, $\overrightarrow{OB}$ = **b**.

Let P divide AB in the ratio $m:n$.

Show that:

1. $\overrightarrow{AP} = \dfrac{m}{m+n}\,\mathbf{b} - \dfrac{m}{m+n}\,\mathbf{a}$

2. $\overrightarrow{OP} = \dfrac{n}{m+n}\,\mathbf{a} + \dfrac{m}{m+n}\,\mathbf{b} = \dfrac{1}{m+n}\,(n\,\mathbf{a} + m\,\mathbf{b})$

G If $\overrightarrow{OA}$ = **a** and $\overrightarrow{OB}$ = **b**, write down the vector **x** = $\overrightarrow{OX}$, in terms of **a** and **b** where X lies on AB and:

1. AX = 2XB

2. 2AX = 3XB

3. 5AX = 2XB

H C = (3, 6) and D = (8, 2).

If E lies on CD and CE : ED = 1 : 3

1. Express $\overrightarrow{OE}$ in terms of $\overrightarrow{OC}$ and $\overrightarrow{OD}$.

2. Calculate the co-ordinates of E.

VECTOR AREAS

1. These parallelograms are defined by the vectors **p** and **q**.

a)

$$\mathbf{p} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\mathbf{q} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

b)

$$\mathbf{p} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$
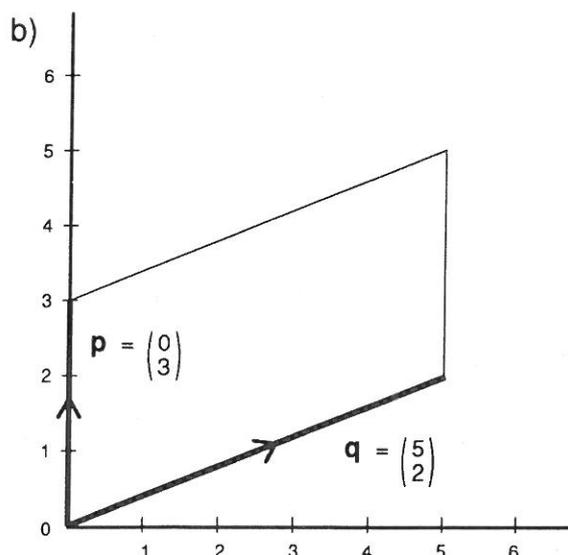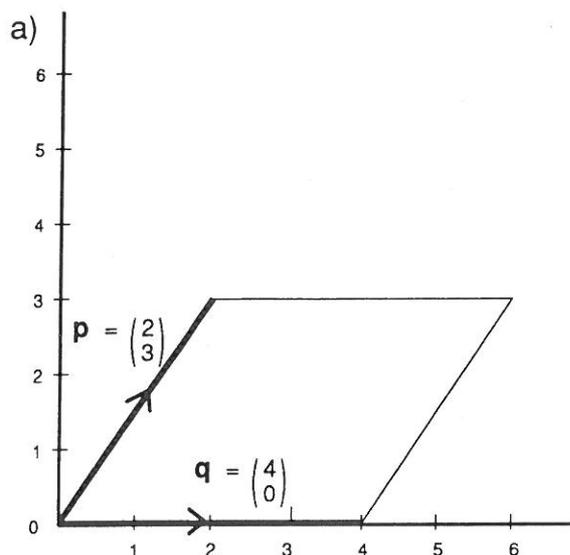
$$\mathbf{q} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

What are their areas?   **Try some others.**

2. In these parallelograms neither vector lies along an axis.

a)

$$\mathbf{p} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\mathbf{q} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

b)

$$\mathbf{q} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\mathbf{p} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$
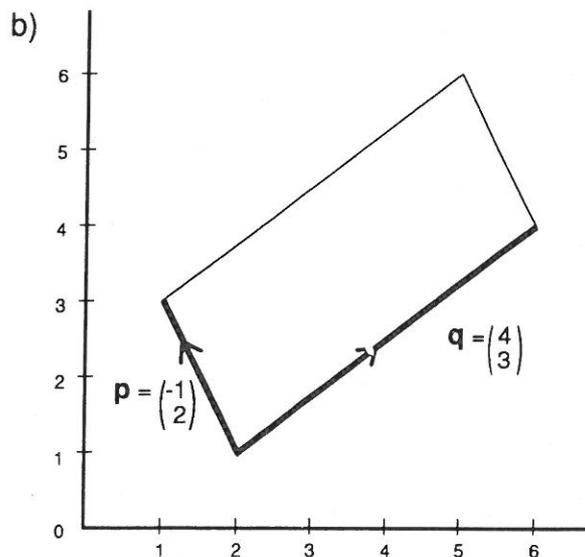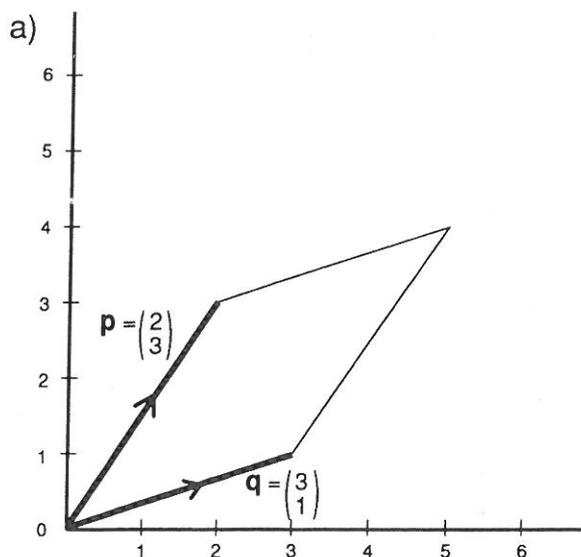
What are their areas?   **Try some others.**

3. Generalise for the area of any parallelogram formed by the vectors $\mathbf{p} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} c \\ d \end{pmatrix}$.